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In acute ΔABC the following relationship holds:

$$bc\sqrt{\cos A}+ca\sqrt{\cos B}+ab\sqrt{\cos C}>\frac{4}{3}R^2$$

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$$\textit{For acute triangle } \sqrt{\cos A}>\cos A\,,$$
$$\textit{therefore}\,,\sum\sqrt{\cos A}>\sum\cos A=1+\frac{r}{R}>1\ (1)$$
$$\textit{We know that for acute triangle } s>2R+r\ (2)$$

$$\left(\begin{array}{c}\textit{since}, \cos A\cos B\cos C=\dfrac{s^2-(2R+r)^2}{4R^2},\\ \textit{for acute triangle } \cos A\cos B\cos C>0 \textit{ then } \dfrac{s^2-(2R+r)^2}{4R^2}>0 \textit{ or } s>2R+r\end{array}\right)$$

$$ab+bc+ca=s^2+r^2+4Rr>s^2\overset{(2)}{>}(2R+r)^2>(2R)^2=4R^2\ (3)$$

$$bc\sqrt{\cos A}+ca\sqrt{\cos B}+ab\sqrt{\cos C}\overset{Chebyshev}{>}$$
$$>\frac{1}{3}(ab+bc+ca)\left(\sum\sqrt{\cos A}\right)\overset{(1)\&(3)}{>}\frac{1}{3}.4R^2.1=\frac{4}{3}R^2$$