ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$\frac{1}{-a+b+c}\sqrt{\sin A}+\frac{1}{a-b+c}\sqrt{\sin B}+\frac{1}{a+b-c}\sqrt{\sin C}>\frac{1}{r\sqrt{3}}$$

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Solution by Tapas Das-India

$$\sum\frac{1}{s-a}=\frac{(s-b)(s-c)+(s-c)(s-a)+(s-a)(s-b)}{(s-a)(s-b)(s-c)}=$$

$$=\frac{3s^2-2s(a+b+c)+(ab+bc+ca)}{sr^2}=\frac{3s^2-4s^2+s^2+r^2+4Rr}{sr^2}=$$

$$=\frac{4R+r}{sr}\overset{Doucet}{\geq}\frac{\sqrt{3}}{r}\ (1)$$

$$0<\sin x<1\ \forall\ x\in\left(0,\frac{\pi}{2}\right),\qquad for\ this\ \sqrt{\sin x}>\sin x$$

$$\sum\sqrt{\sin A}>\sum\sin A\overset{Jordan\ inequality}{>}\sum\frac{2A}{\pi}=2\ (2)$$

$$WLOG\ a\geq b\geq c\ then\ \sin A\geq\sin B\geq\sin C\ \ and\ (s-a)\leq(s-b)\leq(s-c)$$

$$\frac{1}{-a+b+c}\sqrt{\sin A}+\frac{1}{a-b+c}\sqrt{\sin B}+\frac{1}{a+b-c}\sqrt{\sin C}=\sum\frac{1}{-a+b+c}\sqrt{\sin A}=$$

$$=\sum\frac{1}{2(s-a)}\sqrt{\sin A}\overset{Cheycv}{\geq}\frac{1}{3}\cdot\frac{1}{2}\sum\frac{1}{s-a}\cdot\sum\sqrt{\sin A}\overset{(1)\&(2)}{>}\frac{1}{3}\frac{1}{2}\frac{\sqrt{3}}{r}\times 2=\frac{1}{r\sqrt{3}}$$