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In acute ΔABC the following relationship holds:

$$\frac{1}{h_a}\sqrt{\cos A}+\frac{1}{h_b}\sqrt{\cos B}+\frac{1}{h_c}\sqrt{\cos C}\le\frac{1}{r\sqrt{2}}$$

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Solution by Tapas Das-India

$$\sum\sin 2A=(\sin 2A+\sin 2B)+\sin 2C=$$
$$=2\sin(A+B)\cos(A-B)+\sin 2C\overset{A+B+C=\pi}{=}$$

$$=2\sin C\cos(A-B)+2\sin C\cos C=2\sin C\,(\cos(A-B)+\cos C)=$$

$$\overset{A+B+C=\pi}{=}2\sin C\,(\cos(A-B)-\cos(A+B))=4\ \sin A\sin B\sin C\ \ (1)$$

$$\sum\sin A\sin B=\frac{ab+bc+ca}{4R^2}=\frac{s^2+r^2+4Rr}{4R^2}\overset{Gerretsen}{\le}$$
$$\le\frac{4R^2+4Rr+3r^2+r^2+4Rr}{4R^2}=\frac{4(R+r)^2}{4R^2}=\left(1+\frac{R}{r}\right)^2\overset{Euler}{\le}\left(\frac{3}{2}\right)^2\ \ (2)$$

$$\sum\sqrt{\cos A}=\sum\sqrt{\frac{\sin 2A}{2\sin A}}\overset{CBS}{\le}\sqrt{\frac{1}{2}\sum\sin 2A\,.\sum\frac{1}{\sin A}}\overset{(1)}{=}$$
$$=\sqrt{\frac{1}{2}\,4\ \sin A\sin B\sin C\,\frac{\sum\sin A\sin B}{\sin A\sin B\sin C}}\overset{(2)}{\le}\sqrt{2}\cdot\frac{3}{2}$$

$$WLOG\ \ a\ge b\ge c\ \ then\ h_a\le h_b\le h_c\ and\ \cos A\le\cos B\le\cos C$$

$$\frac{1}{h_a}\sqrt{\cos A}+\frac{1}{h_b}\sqrt{\cos B}+\frac{1}{h_c}\sqrt{\cos C}\overset{Chebyshev}{\le}$$
$$\le\frac{1}{3}\sum\frac{1}{h_a}\sum\sqrt{\cos A}\le\frac{1}{3r}.\sqrt{2}.\frac{3}{2}=\frac{1}{r\sqrt{2}}$$

Equality holds for an equilateral triangle.