ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$\frac{a}{b+c}\sqrt{\sin\frac{A}{2}}+\frac{b}{a+c}\sqrt{\sin\frac{B}{2}}+\frac{c}{a+b}\sqrt{\sin\frac{C}{2}}>\frac{1}{\sqrt[4]{2}}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Tapas Das-India

$$We\ know\ that\ f(x)=\sin\frac{x}{2}\ is\ \ increasing\ on\ \left(0,\frac{\pi}{2}\right)$$

$$then\ f(A)=\sqrt{\sin\frac{A}{2}}\overset{A<\frac{\pi}{2}}{<}\sqrt{\sin\frac{\pi}{4}}=\frac{1}{\sqrt[4]{2}}$$

$$\sqrt[4]{2}<\frac{1}{\sqrt{\sin\frac{A}{2}}}\quad or\quad \sqrt[4]{2}\sin\frac{A}{2}<\sqrt{\sin\frac{A}{2}}\ (1)$$

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos\frac{B-C}{2}\sin\frac{B+C}{2}}\overset{A+B+C=\pi\ \&\cos\frac{B-C}{2}<1}{=}\frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\cos\frac{A}{2}}=\sin\frac{A}{2}\ (2)$$

$$\frac{a}{b+c}\sqrt{\sin\frac{A}{2}}+\frac{b}{a+c}\sqrt{\sin\frac{B}{2}}+\frac{c}{a+b}\sqrt{\sin\frac{C}{2}}=\sum\frac{a}{b+c}\sqrt{\sin\frac{A}{2}}\overset{(1)\&(2)}{>}\sum\sqrt[4]{2}\sin^2\frac{A}{2}=$$

$$=\sqrt[4]{2}\sum\sin^2\frac{A}{2}=\frac{\sqrt[4]{2}}{2}\sum2\sin^2\frac{A}{2}=\frac{\sqrt[4]{2}}{2}\sum(1-\cos A)=$$

$$=\frac{\sqrt[4]{2}}{2}\left(3-1-\frac{r}{R}\right)\overset{Euler}{>}\frac{\sqrt[4]{2}}{2}\left(2-\frac{1}{2}\right)=$$

$$=\frac{\sqrt[4]{2}}{2}.\frac{3}{2}=\frac{3\sqrt[4]{2}}{4}\overset{as\ 18>16\ then\ 3\sqrt{2}>4}{>}\frac{3\sqrt[4]{2}}{3\sqrt{2}}=\frac{\sqrt[4]{2}}{\sqrt{2}}=\frac{1}{\sqrt[4]{2}}$$