ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $\mathrm{r}:\mathrm{r}_a:\mathrm{R}=2:12:5$, the fo$ll$owing re$la$tionship ho$l$ds :

$$\left(a^4+\mathrm{b}^4+\mathrm{c}^4\right)\left(\frac{1}{a^4}+\frac{1}{\mathrm{b}^4}+\frac{1}{\mathrm{c}^4}\right)\geq\frac{100}{9}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\mathrm{r}}{\mathrm{r}_a}=\frac{1}{6}\Rightarrow\frac{\mathrm{s}-a}{\mathrm{s}}=\frac{1}{6}\Rightarrow 5\mathrm{s}=6a\Rightarrow a\overset{①}{=}\frac{5\mathrm{s}}{6}\Rightarrow 12a=5(a+\mathrm{b}+\mathrm{c})$$

$$\Rightarrow\mathrm{b}+\mathrm{c}=\frac{7a}{5}\overset{\text{via } ①}{=}\frac{7}{5}\cdot\frac{5\mathrm{s}}{6}\Rightarrow\mathrm{b}+\mathrm{c}\overset{②}{=}\frac{7\mathrm{s}}{6}\ \text{and } also,\frac{\mathrm{r}}{\mathrm{R}}=\frac{2}{5}\Rightarrow\frac{5\mathrm{F}}{\mathrm{s}}=\frac{2a\mathrm{bc}}{4\mathrm{F}}$$

$$\Rightarrow 10\mathrm{s}(\mathrm{s}-a)(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})=\mathrm{s}a\mathrm{bc}\overset{\text{via } ①}{\Rightarrow}10.\frac{\mathrm{s}}{6}.\left(-\mathrm{s}^2+\mathrm{s}.\frac{5\mathrm{s}}{6}+\mathrm{bc}\right)=\frac{5\mathrm{s}}{6}.\mathrm{bc}$$

$$\Rightarrow 2\left(\mathrm{bc}-\frac{\mathrm{s}^2}{6}\right)=\mathrm{bc}\Rightarrow\mathrm{bc}\overset{③}{=}\frac{\mathrm{s}^2}{3}\ \therefore ②\ \text{and}\ ③\Rightarrow\mathrm{b}+\frac{\mathrm{s}^2}{3\mathrm{b}}=\frac{7\mathrm{s}}{6}$$

$$\Rightarrow 6\mathrm{b}^2-7\mathrm{bs}+2\mathrm{s}^2=0\Rightarrow(\mathrm{s}-2\mathrm{b})(2\mathrm{s}-3\mathrm{b})=0$$

$$\Rightarrow\mathrm{b}=\frac{\mathrm{s}}{2}\left(\text{with c}=\frac{2\mathrm{s}}{3}\right)\text{ or b}=\frac{2\mathrm{s}}{3}\left(\text{with c}=\frac{\mathrm{s}}{2}\right)\Rightarrow\text{in either } case:$$

$$\mathrm{b}^4+\mathrm{c}^4\overset{(*)}{=}\left(\frac{\mathrm{s}}{2}\right)^4+\left(\frac{2\mathrm{s}}{3}\right)^4\ and\ \frac{1}{\mathrm{b}^4}+\frac{1}{\mathrm{c}^4}\overset{(**)}{=}\frac{16+\left(\frac{3}{2}\right)^4}{\mathrm{s}^4}\ \therefore ①,(*)\ and\ (**)\Rightarrow$$

$$\left(a^4+\mathrm{b}^4+\mathrm{c}^4\right)\left(\frac{1}{a^4}+\frac{1}{\mathrm{b}^4}+\frac{1}{\mathrm{c}^4}\right)=\left(\left(\frac{5\mathrm{s}}{6}\right)^4+\left(\frac{\mathrm{s}}{2}\right)^4+\left(\frac{2\mathrm{s}}{3}\right)^4\right)\left(\frac{\left(\frac{6}{5}\right)^4+16+\left(\frac{3}{2}\right)^4}{\mathrm{s}^4}\right)$$

$$\approx 17.173556>\frac{100}{9}\ \therefore\left(a^4+\mathrm{b}^4+\mathrm{c}^4\right)\left(\frac{1}{a^4}+\frac{1}{\mathrm{b}^4}+\frac{1}{\mathrm{c}^4}\right)\geq\frac{100}{9}\ \forall\,\Delta\,\text{ABC}$$

with $\mathrm{r}:\mathrm{r}_a:\mathrm{R}=2:12:5$ (QED)