ROMANIAN MATHEMATICAL MAGAZINE

In *any* Δ ABC the *following* *relationship* *holds* :

$$\frac{1}{9r^2} \geq \sum_{cyc} \frac{1}{2r_a^2 + r_b r_c} \geq \frac{4R + r}{s^2(4R - 5r)}$$

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$$\sum_{cyc}\left((2r_b^2 + r_c r_a)(2r_c^2 + r_a r_b)\right) =$$

$$= 4\sum_{cyc} r_b^2 r_c^2 + 2\sum_{cyc}\left(r_b r_c\left(\sum_{cyc} r_a^2 - r_a^2\right)\right) + r_a r_b r_c \sum_{cyc} r_a$$

$$= 4\left(s^4 - 2rs^2(4R + r)\right) + 2s^2((4R + r)^2 - 2s^2) - 2rs^2(4R + r) + rs^2(4R + r) \therefore$$

$$\sum_{cyc}\left((2r_b^2 + r_c r_a)(2r_c^2 + r_a r_b)\right) = s^2(4R + r)(8R - 7r) \rightarrow (1)$$

$$\textit{and} \prod_{cyc}(2r_a^2 + r_b r_c) = 2r_a r_b r_c \sum_{cyc} r_a^3 + 4\sum_{cyc} r_b^3 r_c^3 + 9r_a^2 r_b^2 r_c^2$$

$$= 2rs^2((4R + r)^3 - 12Rs^2) + 4(s^6 - 12Rrs^4) + 9r^2 s^4$$

$$= s^2\left(4s^4 - 72Rrs^2 + \frac{2(4R + r)^2}{3}.(12Rr + 3r^2) + 9r^2 s^2\right) \overset{\text{Gerretsen + Euler}}{\leq}$$

$$s^2\left(4s^4 - 72Rrs^2 + \frac{2s^2(4R + r)^2}{3} + 9r^2 s^2\right) \therefore \sum_{cyc} \frac{1}{2r_a^2 + r_b r_c} =$$

$$\frac{\sum_{cyc}\left((2r_b^2 + r_c r_a)(2r_c^2 + r_a r_b)\right)}{\prod_{cyc}(2r_a^2 + r_b r_c)} \geq \frac{\sum_{cyc}\left((2r_b^2 + r_c r_a)(2r_c^2 + r_a r_b)\right)}{s^4\left(4s^2 - 72Rr + \frac{2(4R+r)^2}{3} + 9r^2\right)}$$

$$\overset{\text{via (1)}}{=} \frac{3(4R + r)(8R - 7r)}{s^2(12s^2 - 216Rr + 2(4R + r)^2 + 27r^2)} \overset{\text{Gerretsen}}{\geq}$$

$$\frac{3(4R + r)(8R - 7r)}{s^2(12(4R^2 + 4Rr + 3r^2) - 216Rr + 2(4R + r)^2 + 27r^2)} \overset{?}{\geq} \frac{4R + r}{s^2(4R - 5r)}$$

$$\Leftrightarrow 4(4R - 5r)(R - 2r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \therefore \sum_{cyc} \frac{1}{2r_a^2 + r_b r_c} \geq$$

$$\frac{4R + r}{s^2(4R - 5r)} \textit{ and also}, \sum_{cyc} \frac{1}{2r_a^2 + r_b r_c} \overset{\text{AM-GM}}{\leq} \sum_{cyc} \frac{1}{3r_a.\sqrt[3]{r_a r_b r_c}} = \frac{1}{3r.\sqrt[3]{rs^2}} \overset{\text{Mitrinovic}}{\leq}$$

$$\frac{1}{3r.\sqrt[3]{27r^3}} \therefore \sum_{cyc} \frac{1}{2r_a^2 + r_b r_c} \leq \frac{1}{9r^2} \textit{ and so}, \frac{1}{9r^2} \geq \sum_{cyc} \frac{1}{2r_a^2 + r_b r_c} \geq \frac{4R + r}{s^2(4R - 5r)}$$

$\forall \Delta$ ABC,'' = '' iff Δ ABC is equi*lateral* (QED)