ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$(m_b+m_c)\tan\frac{A}{2}+(m_c+m_a)\tan\frac{B}{2}+(m_a+m_b)\tan\frac{C}{2}\ge 12\sqrt{3}r\left(\sum(\cos A)-1\right)$$

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Lemma 1:

$$\prod\tan\frac{A}{2}=\prod\frac{r_a}{s}$$

Proof:

$$\tan\frac{A}{2}=\sqrt{\frac{(p-b)(p-c)}{p(p-a)}}=\sqrt{\frac{p(p-a)(p-b)(p-c)}{p^2(p-a)^2}}=\frac{F}{p(p-a)}=\frac{r_a}{p}$$

$$\tan\frac{B}{2}=\frac{r_b}{p},\ \tan\frac{C}{2}=\frac{r_c}{p}\Rightarrow\ \prod\tan\frac{A}{2}=\frac{r_a}{p}\cdot\frac{r_b}{p}\cdot\frac{r_c}{p}=\frac{r_ar_br_c}{p^3}$$

Lemma 2:

$$\sum\cos A\le\frac{3}{2}$$

Proof:

$$\sum\cos A=1+4\cdot\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$$

$$Jensen\,Theorem\ \Rightarrow\frac{\sum\sin\frac{A}{2}}{3}\le\sin\left(\frac{A+B+C}{2\cdot3}\right)=\frac{1}{2}$$

$$\left(\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}\right)^{\frac{1}{3}}\le\frac{\left(\sum\sin\frac{A}{2}\right)}{3}\le\frac{1}{2}\ \Rightarrow\ \cos A\le\frac{3}{2}$$

$$(m_b+m_c)\tan\frac{A}{2}+(m_c+m_a)\tan\frac{B}{2}+(m_a+m_b)\tan\frac{C}{2}\ge$$

$$\ge3\sqrt[3]{(m_a+m_b)(m_b+m_c)(m_a+m_c)\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}$$

$$\geq 3\sqrt[3]{8m_am_bm_c\cdot\frac{r_ar_br_c}{p^3}}\geq 6\sqrt[3]{\frac{(r_ar_br_c)^2}{p^3}}=6\sqrt[3]{\frac{F^2p^2}{p^3}}=6\sqrt[3]{sr^2}\geq 6\sqrt[3]{3\sqrt{3}r^3}=6\sqrt{3}r=$$

$$=12\sqrt{3}r\cdot\frac{1}{2}\geq 12\sqrt{3}r\left(\sum(\cos A)-1\right)$$