ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$r_b r_c \tan\frac{A}{2} + r_c r_a \tan\frac{B}{2} + r_a r_b \tan\frac{C}{2} \geq 9\sqrt{3}r^2$$

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$Lemma$:

$$\tan\frac{A}{2} = \frac{r}{p-a}$$

$Proof$:

$$Let\ I - incenter,\ ID \perp AC\ and\ ID = r,\ AD = p-a.$$

$$\Delta IDA\ \Rightarrow\ \tan\frac{A}{2} = \frac{r}{AD} = \frac{r}{p-a}$$

$$r_a = \frac{pr}{p-a},\ r_b = \frac{pr}{p-b},\ r_c = \frac{pr}{p-c}\ and\ \tan\frac{A}{2} = \frac{r}{p-a}$$

$$r_b r_c \tan\frac{A}{2} + r_c r_a \tan\frac{B}{2} + r_a r_b \tan\frac{C}{2} = \sum r_b r_c \tan\frac{A}{2} = \sum \frac{pr}{p-b}\cdot\frac{pr}{p-c}\cdot\frac{r}{p-a} =$$

$$= \sum \frac{p\cdot p^2 r^3}{p(p-a)(p-b)(p-c)} = \sum \frac{p^3 r^3}{p^2 r^2} = \sum pr = 3pr \geq 9\sqrt{3}r^2$$

$$Equality\ holds\ for:\ a = b = c.$$