ROMANIAN MATHEMATICAL MAGAZINE

If I −incenter in ΔABC then:

$$\frac{AI^2}{b^2+c^2}+\frac{BI^2}{a^2+c^2}+\frac{CI^2}{a^2+b^2}\le\frac{1}{2}$$

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Solution by Ertan Yildirim-Turkiye

$Lemma$:

$$AI^2=bc-4rR$$

$Proof$:

$$AI^2=\frac{r^2}{\sin^2\frac{A}{2}}=\frac{r^2}{\frac{(p-b)(p-c)}{bc}}=r^2\cdot\frac{bc}{(p-b)(p-c)}=$$

$$=r^2\cdot bc\cdot\frac{p(p-a)}{p(p-a)(p-b)(p-c)}=\frac{r^2bc\cdot p\cdot(p-a)}{p^2r^2}=$$

$$=bc\cdot\left(1-\frac{a}{p}\right)=bc-\frac{abc}{p}=bc-\frac{4Rrp}{p}=bc-4rR$$

$$\sum\frac{AI^2}{b^2+c^2}\le\sum\frac{bc-4rR}{2bc}=\sum\frac{1}{2}-\frac{4rR}{2bc}=\frac{3}{2}-2rR\cdot\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)=$$

$$=\frac{3}{2}-2rR\cdot\frac{a+b+c}{abc}=\frac{3}{2}-2Rr\cdot\frac{2p}{4rRp}=\frac{3}{2}-1=\frac{1}{2}$$

$$Equality\ holds\ for\ a=b=c.$$