ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{w_a+w_b}{w_c}}+\frac{1013R^3}{4r^3}\geq 2026+\sum\sqrt{\frac{m_a+m_b}{m_c}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\frac{w_a+w_b}{w_c}\overset{AM-GM}{\geq}\frac{2\sqrt{w_aw_b}}{w_c}\ (1)$$

$$\sum\sqrt{\frac{w_a+w_b}{w_c}}\overset{(1)}{\geq}\sum\frac{\sqrt{2}\sqrt[4]{w_aw_b}}{\sqrt{w_c}}\overset{AM-GM}{\geq}3\sqrt{2}\sqrt[3]{\frac{\sqrt{w_aw_bw_c}}{\sqrt{w_aw_bw_c}}}=3\sqrt{2}\ (2)$$

$$\sum\sqrt{\frac{m_a+m_b}{m_c}}\overset{Panatpal}{\leq}\sum\sqrt{\frac{\frac{Rh_a}{2r}+\frac{Rh_b}{2r}}{m_c}}\overset{m_c\geq h_c}{\leq}\sum\sqrt{\frac{\frac{Rh_a}{2r}+\frac{Rh_b}{2r}}{h_c}}=$$

$$=\sqrt{\frac{R}{2r}}\sum\sqrt{\frac{h_a}{h_c}+\frac{h_b}{h_c}}=\sqrt{\frac{R}{2r}}\sum\sqrt{\frac{c}{a}+\frac{c}{b}}\leq$$

$$\overset{CBS}{\leq}\sqrt{\frac{R}{2r}}\sqrt{3\left(\sum\frac{c}{a}+\frac{c}{b}\right)}=\sqrt{\frac{R}{2r}}\sqrt{3\left(\sum\frac{a}{b}+\frac{a}{a}\right)}\overset{Bandila}{\leq}\sqrt{\frac{R}{2r}}\sqrt{\frac{3.3R}{r}}=\frac{3\sqrt{2}}{2}\frac{R}{r}\ (3)$$

We need to show:

$$\sum\sqrt{\frac{w_a+w_b}{w_c}}+\frac{1013R^3}{4r^3}\geq 2026+\sum\sqrt{\frac{m_a+m_b}{m_c}}$$

$$3\sqrt{2}+\frac{1013R^3}{4r^3}\geq 2026+\frac{3\sqrt{2}}{2}\frac{R}{r}$$

$$1013x^3-6\sqrt{2}x+12\sqrt{2}-8104\overset{\frac{R}{r}=x\geq 2\ (Euler)}{\geq}0$$

$$1013(x^3-8)-6\sqrt{2}(x-2)\geq 0$$

$$(x-2)\big(1013(x^2+2x+4)-6\sqrt{2}\big)\geq 0\ true\ as\ x\geq 2$$

Equality holds for an equilateral triangle.