ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{\sum_{cyc} m_a^2}{\sum_{cyc} n_a^2}\cdot\frac{27r^2}{s^2}+\frac{R^4}{16r^4}\ge 2$$

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$$\frac{\sum_{cyc} m_a^2}{\sum_{cyc} n_a^2}\cdot\frac{27r^2}{s^2}+\frac{R^4-32r^4}{16r^4}\overset{\text{Bogdan Fustei}}{=}\frac{\frac{3}{2}(s^2-4Rr-r^2)}{\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}\cdot\frac{27r^2}{s^2}+$$

$$\frac{R^4-32r^4}{16r^4}=\frac{\frac{3}{2}(s^2-4Rr-r^2)}{3s^2-\frac{r}{R}(s^2+(4R+r)^2)}\cdot\frac{27r^2}{s^2}+\frac{R^4-32r^4}{16r^4}\overset{?}{\ge}0$$

$$\Leftrightarrow\frac{8r^6.81R(s^2-4Rr-r^2)+(R^4-32r^4)s^2\left((3R-r)s^2-r(4R+r)^2\right)}{16r^4s^2\left((3R-r)s^2-r(4R+r)^2\right)}\overset{?}{\ge}0$$

$$\Leftrightarrow\left(3R^5-R^4r-96Rr^4+32r^5\right)s^4-$$

$$r\left(16R^6+8R^5r+R^4r^2-512R^2r^4-904Rr^5-32r^6\right)s^2-648Rr^7(4R+r)\boxed{\overset{?}{\underset{(*)}{\ge}}}0$$

$\boxed{\text{Case 1}}$ $3R^5-R^4r-96Rr^4+32r^5\ge 0\Leftrightarrow(3R-r)(R^4-32r^4)\ge 0\Leftrightarrow R^4-32r^4\ge 0$ *and* then $\because P=\left(3R^5-R^4r-96Rr^4+32r^5\right)(s^2-16Rr+5r^2)^2+\left(80R^6-70R^5r+9R^4r^2-2560R^2r^4+2888Rr^5-288r^6\right)(s^2-16Rr+5r^2)$

$$\overset{\text{Gerretsen}}{\ge}0\left(\begin{matrix}\because 80R^6-70R^5r+9R^4r^2-2560R^2r^4+2888Rr^5-288r^6\\=(80R^2-70Rr+9r^2)(R^4-32r^4)+648Rr^5\end{matrix}\right)$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\ge}P\Leftrightarrow$

$$512t^7-784t^6+259t^5-20t^4-16384t^3+32864t^2-12176t+640\overset{?}{\ge}0$$

$$\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left((t-2)\left(\begin{matrix}512t^5+1264t^4+3267t^3+\\7992t^2+2516t+10960\end{matrix}\right)+21600\right)\overset{?}{\ge}0\to\text{true}$$

$$\because t\overset{\text{Euler}}{\ge}2\Rightarrow(*)\text{ is true}$$

$\boxed{\text{Case 2}}$ $3R^5-R^4r-96Rr^4+32r^5<0$ *and* then $\because Q=\left(3R^5-R^4r-96Rr^4+32r^5\right)\left(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\right)\overset{\text{Double-Rouche}}{\ge}0\ \therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\ge}Q\Leftrightarrow$

$$\left(12R^7+40R^6r-34R^5r^2+R^4r^3-384R^3r^4-1280R^2r^5+1736Rr^6-32r^7\right)s^2$$

$$=\left(F(R,r)\right)s^2\boxed{\overset{?}{\underset{(**)}{\ge}}}r\left(\begin{matrix}192R^8+80R^7r-12R^6r^3-9R^5r^4-\\6145R^4r^5-2560R^3r^6+2976R^2r^6+936Rr^7+32r^8\end{matrix}\right)$$

$\boxed{Case\ 2i}$ $\mathbf{F(R,r) \geq 0}$ ***and then*** $\because \left(F(R,r)\right)(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\geq} 0$ **∴ in order to prove** $(**)$**, it suffices to prove :** **LHS of** $(**) \overset{?}{\geq} \left(F(R,r)\right)(s^2-16Rr+5r^2)$

$$\Leftrightarrow 500t^7-732t^6+195t^5-4t^4-16000t^3+31200t^2-10128t+128 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}500t^5+1268t^4+3267t^3+7992t^2+\\2900t+10832\end{pmatrix}+21600\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true}$$

$\boxed{Case\ 2ii}$ $\mathbf{F(R,r) < 0}$ ***and then*** $\because \left(F(R,r)\right)(s^2-4R^2-4Rr-3r^2) \overset{\text{Gerretsen}}{\geq} 0$

∴ in order to prove $(**)$**, it suffices to prove :** **LHS of** $(**) \overset{?}{\geq} \left(F(R,r)\right)(s^2-4R^2-4Rr-3r^2)$

$$\Leftrightarrow 48t^9+16t^8-20t^7-1625t^5-508t^4+3232t^3+4144t-128 \overset{?}{\geq} 0 \Leftrightarrow$$

$$(t-2)\left((t-2)\begin{pmatrix}48t^7+208t^6+620t^5+1648t^4+2487t^3+\\2848t^2+4676t+7312\end{pmatrix}+14688\right) \overset{?}{\geq} 0$$

$\rightarrow$ **true** $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**)$ **is true ∴ combining** ***all*** **cases,** $(*)$ **is true** $\forall\, \Delta\, ABC$**,** $'' = ''$ **iff** $\Delta\, ABC$ **is equilateral (QED)**