ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{3\sqrt[3]{m_a m_b m_c}}{m_a+m_b+m_c}+253\frac{R^3}{r^3}\geq 2025$$

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Solution by Tapas Das-India

$$m_a m_b m_c \geq \sqrt{s(s-a)}\sqrt{s(s-b)}\sqrt{s(s-c)}=\sqrt{s^3 sr^2}=$$
$$=s^2 r \overset{Mitrinovic}{\geq} 27r^3 \text{ and } \sqrt[3]{m_a m_b m_c}\geq 3r \ (1)$$

$$m_a+m_b+m_c \overset{Gotman-II}{\leq} \frac{9R}{2}(2) \text{ and } \frac{3\sqrt[3]{m_a m_b m_c}}{m_a+m_b+m_c}\overset{(1)\&(2)}{\geq}\frac{2r}{R}$$

We need to show:

$$\frac{3\sqrt[3]{m_a m_b m_c}}{m_a+m_b+m_c}+253\frac{R^3}{r^3}\geq 2025$$

$$\frac{2r}{R}+253\frac{R^3}{r^3}\geq 2025$$

$$253x^4-2025x+2\overset{\frac{R}{r}=x\geq 2}{\geq} 0$$

$$(x-2)(253x^3+506x^2+1012x-1)\geq 0 \text{ true as } x\geq 2$$

Equality holds for an equilateral triangle.