ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC holds :

$$\min\left\{\sum_{cyc}\frac{m_a+m_b}{m_b+m_c},\sum_{cyc}\frac{a+b}{b+c}\right\}+\frac{R^5}{32r^5}\ge 1+\max\left\{\sum_{cyc}\frac{m_a^2+m_b^2}{m_b^2+m_c^2},\sum_{cyc}\frac{a^2+b^2}{b^2+c^2}\right\}$$

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$$\text{Via Bergstrom}, \forall\, x,y,z>0, \sum_{cyc}\frac{x}{y}\ge\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}xy}\text{ and substituting } x\equiv s-a,$$

$$y\equiv s-b, z\equiv s-c, \text{we get}: \left(\sum_{cyc}(s-a)(s-b)\right)\sum_{cyc}(s-a)(s-b)^2\ge$$

$$\prod_{cyc}(s-a).\left(\sum_{cyc}(s-a)\right)^2\Rightarrow(4Rr+r^2)\sum_{cyc}\Big((s-b)(-s^2+sc+ab)\Big)\ge r^2s.s^2$$

$$\Rightarrow\left(-3s^3+s^2(2s)+s\sum_{cyc}ab+s^2(2s)-s\sum_{cyc}bc-\sum_{cyc}ab^2\right)\ge\frac{rs^3}{4R+r}\Rightarrow$$

$$s^3-\sum_{cyc}ab^2\ge\frac{rs^3}{4R+r}\Rightarrow\sum_{cyc}ab^2\le\frac{4Rs^3}{4R+r}\Rightarrow\sum_{cyc}\frac{ab^2}{abc}\le\frac{4Rs^3}{4Rrs(4R+r)}$$

$$\Rightarrow\sum_{cyc}\frac{c}{a}\overset{(1)}{\le}\frac{s^2}{r(4R+r)}\text{ and again, via Bergstrom}, \forall\, x,y,z>0, \sum_{cyc}\frac{y}{x}\ge\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}xy}$$

$$\text{and substituting } x\equiv s-a, y\equiv s-b, z\equiv s-c, \text{we get}:$$

$$\left(\sum_{cyc}(s-a)(s-b)\right)\sum_{cyc}(s-a)^2(s-b)\ge(s-a)(s-b)(s-c).\left(\sum_{cyc}(s-a)\right)^2$$

$$\Rightarrow s^3-\sum_{cyc}a^2b\ge\frac{rs^3}{4R+r}\Rightarrow\sum_{cyc}a^2b\le\frac{4Rs^3}{4R+r}\Rightarrow\sum_{cyc}\frac{a^2b}{abc}\le\frac{4Rs^3}{4Rrs(4R+r)}$$

$$\Rightarrow\sum_{cyc}\frac{a}{c}\overset{(2)}{\le}\frac{s^2}{r(4R+r)}\ \&\ \frac{a^2+b^2}{b^2+c^2}\overset{\substack{\text{Tereshin}\\\text{and}\\\text{AM-GM}}}{\le}\frac{4R.m_c}{2bc}=\frac{m_c}{h_a}=\frac{am_c}{2F}\ \&\text{ implementing it on}$$

$$\text{triangle with sides }\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3};\ \text{area via trivial calculations}=\frac{F}{3},\text{we get}:$$

$$\frac{\frac{4}{9}\left(m_a^2+m_b^2\right)}{\frac{4}{9}\left(m_b^2+m_c^2\right)} \geq \frac{\frac{2m_a}{3}\cdot\frac{c}{2}}{\frac{2F}{3}} = \frac{cm_a}{2F} \overset{Panaitopol}{\leq} \frac{c}{2rs}\cdot\frac{Rs}{a} = \frac{R}{2r}\cdot\frac{c}{a} \Rightarrow \frac{m_a^2+m_b^2}{m_b^2+m_c^2} \leq \frac{R}{2r}\cdot\frac{c}{a} \therefore$$

$$\sum_{cyc}\frac{m_a^2+m_b^2}{m_b^2+m_c^2} \leq \frac{R}{2r}\cdot\sum_{cyc}\frac{c}{a} \overset{via\ ①}{\leq} \frac{R}{2r}\cdot\frac{s^2}{r(4R+r)} \overset{Gerretsen}{\leq} \frac{R}{2r}\cdot\frac{4R^2+4Rr+3r^2}{r(4R+r)} =$$

$$\frac{t(16t^2+8t+3)}{8t+1}\left(t=\frac{R}{2r}\right) \overset{?}{\leq} 3+t^5-1 \Leftrightarrow 8t^6+t^5-16t^3-8t^2+13t+2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-1)\Big((t-1)(8t^4+17t^3+26t^2+19t+4)+2\Big) \to true \because t \overset{Euler}{\geq} 1$$

$$\therefore \sum_{cyc}\frac{m_a^2+m_b^2}{m_b^2+m_c^2} - \frac{R^5}{32r^5} + 1 \overset{③}{\leq} 3 \textit{ and also}, \sum_{cyc}\frac{m_a+m_b}{m_b+m_c}, \sum_{cyc}\frac{a+b}{b+c} \overset{AM-GM}{\geq} 3$$

$$\Rightarrow \min\left\{\sum_{cyc}\frac{m_a+m_b}{m_b+m_c}, \sum_{cyc}\frac{a+b}{b+c}\right\} \geq 3 \overset{via\ ③}{\geq} \sum_{cyc}\frac{m_a^2+m_b^2}{m_b^2+m_c^2} - \frac{R^5}{32r^5} + 1$$

$$\therefore \min\left\{\sum_{cyc}\frac{m_a+m_b}{m_b+m_c}, \sum_{cyc}\frac{a+b}{b+c}\right\} + \frac{R^5}{32r^5} \boxed{\overset{(*)}{\geq}} 1 + \sum_{cyc}\frac{m_a^2+m_b^2}{m_b^2+m_c^2} \textit{ and also},$$

$$\sum_{cyc}\frac{a^2+b^2}{b^2+c^2} \overset{\substack{Bandila\\and\\AM-GM}}{\leq} \sum_{cyc}\frac{\frac{R}{r}.ab}{2bc} = \frac{R}{2r}\cdot\sum_{cyc}\frac{a}{c} \overset{via\ ②}{\leq} \frac{s^2}{r(4R+r)} \overset{Gerretsen}{\leq}$$

$$\frac{R}{2r}\cdot\frac{4R^2+4Rr+3r^2}{r(4R+r)} \overset{shown\ earlier}{\leq} 3 + \frac{R^5}{32r^5} - 1 \overset{earlier}{\leq} \min\left\{\sum_{cyc}\frac{m_a+m_b}{m_b+m_c}, \sum_{cyc}\frac{a+b}{b+c}\right\}$$

$$+\frac{R^5}{32r^5} - 1 \therefore \min\left\{\sum_{cyc}\frac{m_a+m_b}{m_b+m_c}, \sum_{cyc}\frac{a+b}{b+c}\right\} + \frac{R^5}{32r^5} \boxed{\overset{(**)}{\geq}} 1 + \sum_{cyc}\frac{a^2+b^2}{b^2+c^2}$$

$\therefore (*), (**) \Rightarrow$ **proposed inequality is true** $\forall\ \Delta\ ABC$,

$'' ='' $ **iff** Δ **ABC is equilateral (QED)**