ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^3+\sum_{cyc}\left(\frac{a}{b+c}\right)^4+\left(\frac{R}{2r}\right)^5\ge 1+\sum_{cyc}\left(\frac{a}{b+c}\right)^3+\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^4$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(b+c)^2\overset{?}{\ge}32Rr\cos^2\frac{A}{2}=8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$
$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)\Leftrightarrow$$
$$(b+c)^2+4a^2-4a(b+c)\overset{?}{\ge}0\Leftrightarrow(b+c-2a)^2\overset{?}{\ge}0\to\text{true}$$
$$\therefore b+c\ge\sqrt{32Rr}.\cos\frac{A}{2}\ \textit{and analogs}\to(m)\ \textit{and}\ \sum_{cyc}\frac{b+c}{b+c-a}=\frac{1}{2}\sum_{cyc}\frac{s+s-a}{s-a}$$
$$=\frac{1}{2}\left(\frac{s(4Rr+r^2)}{r^2s}+3\right)=\frac{2R+2r}{r}=4.\frac{R}{2r}+2=\frac{abc\left(\sum_{cyc}a\right)}{4F^2}+2\ \therefore\ \sum_{cyc}\frac{b+c}{b+c-a}$$
$$=\frac{abc\left(\sum_{cyc}a\right)}{4F^2}+2\ \text{and implementing it ton } a \text{ triangle with sides }\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}$$
$$\text{whose area via of trivial calculations}=\frac{F}{3},\text{we get}:\sum_{cyc}\frac{m_b+m_c}{m_b+m_c-m_a}=$$
$$\frac{\frac{16}{81}m_am_bm_c\left(\sum_{cyc}m_a\right)}{\frac{4F^2}{9}}+2\le\frac{4\sum_{cyc}m_a^2m_b^2}{9r^2s^2}+2=\frac{4.\frac{9}{16}\sum_{cyc}a^2b^2}{9r^2s^2}+2\overset{\text{Goldstone}}{\le}$$
$$\frac{4R^2s^2}{4r^2s^2}+2\ \therefore\ \sum_{cyc}\frac{m_b+m_c}{m_b+m_c-m_a}\le\frac{R^2}{r^2}+2\to (1)\ \textit{and now,}$$

$$\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^3-\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^4=\sum_{cyc}\frac{\left(\frac{m_a}{m_b+m_c}\right)^3}{\frac{m_b+m_c}{m_b+m_c-m_a}}\overset{\text{Holder}}{\ge}\frac{\left(\sum_{cyc}\frac{m_a}{m_b+m_c}\right)^3}{\sum_{cyc}\frac{m_b+m_c}{m_b+m_c-m_a}}\overset{\substack{\text{Nesbitt}\\ \text{and}\\ \text{via (1)}}}{\ge}$$
$$\frac{27}{24\left(\frac{R^2}{r^2}+2\right)}\ \therefore\ \sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^3-\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^4\ge\frac{9}{8(4t^4+2)}\to (2)\ \left(t=\sqrt{\frac{R}{2r}}\right)$$
$$\text{Also,}\sum_{cyc}\left(\frac{a}{b+c}\right)^3-\sum_{cyc}\left(\frac{a}{b+c}\right)^4\overset{\text{Chebyshev}}{\le}\sum_{cyc}\left(\frac{a}{b+c}\right)^3-\frac{1}{3}.\sum_{cyc}\left(\frac{a}{b+c}\right)^3.\sum_{cyc}\frac{a}{b+c}$$
$$\overset{\text{Nesbitt}}{\le}=\frac{1}{2}.\sum_{cyc}\left(\frac{a}{b+c}\right)^3\overset{\text{via (m)}}{\le}\frac{1}{2}.\sum_{cyc}\left(\frac{4R\cos\frac{A}{2}\sin\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}}\right)^3$$
$$=\frac{1}{2}.\frac{R}{2r}.\sqrt{\frac{R}{2r}}.\left(\left(\sum_{cyc}\sin\frac{A}{2}\right)^3-3\prod_{cyc}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)\right)\overset{\substack{\text{Jensen}\\ \text{and}\\ \text{Cesaro}}}{\le}\frac{1}{2}.t^3\left(\frac{27}{8}-24.\frac{r}{4R}\right)$$
$$=t^3\left(\frac{27}{16}-\frac{3}{2t}\right)\overset{?}{\le}\frac{9}{8(4t^4+2)}+t^{10}-1$$

$$\Leftrightarrow 32t^{14}+16t^{10}-54t^{7}+48t^{6}-32t^{4}-27t^{3}+24t^{2}-7\overset{?}{\geq}0\Leftrightarrow$$

$$(t-1)\left(\begin{matrix}32t^{13}+32t^{12}+32t^{11}+32t^{10}+48t^{9}+48t^{8}+42t^{7}+6t^{6}(t-1)+\\42t^{5}+42t^{4}+t(t-1)(7(t-1)+3t)+7\end{matrix}\right)\overset{?}{\geq}0$$

$$\to \text{true} \because t\overset{\text{Euler}}{\geq}1 \therefore \sum_{cyc}\left(\frac{a}{b+c}\right)^{3}-\sum_{cyc}\left(\frac{a}{b+c}\right)^{4}\leq\frac{9}{8(4t^{4}+2)}+t^{10}-1\overset{\text{via ②}}{\leq}$$

$$\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^{3}-\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^{4}+\left(\frac{R}{2r}\right)^{5}-1\therefore\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^{3}+\sum_{cyc}\left(\frac{a}{b+c}\right)^{4}+$$

$$\left(\frac{R}{2r}\right)^{5}\geq 1+\sum_{cyc}\left(\frac{a}{b+c}\right)^{3}+\sum_{cyc}\left(\frac{m_a}{m_b+m_c}\right)^{4},''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$