ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum\left(\frac{b+c}{a}\right)^2+\frac{R^4}{r^4}\geq\sum\left(\frac{m_a+m_b}{m_c}\right)^2+16$$

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$$We\ know\ that\ a^2+b^2=(a+b)^2-2ab\ \ (1)$$

$$\sum\frac{1}{s(s-c)}=\frac{\left((s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)\right)}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{3s^2-2s(a+b+c)+ab+bc+ca}{s^2r^2}=\frac{3s^2-2s.(2s)+s^2+r^2+4Rr}{s^2r^2}=$$

$$=\frac{r(4R+r)}{s^2r^2}\overset{s^2\geq 3r(4R+r)}{\leq}\frac{r(4R+r)}{3r(4R+r)r^2}=\frac{1}{3r^2}\ (2)$$

$$\sum\left(\frac{m_a+m_b}{m_c}\right)^2+16=\sum\left(\left(\frac{m_a+m_b}{m_c}\right)^2+(1)^2\right)+13\overset{(1)}{=}$$

$$=\sum\left(\left(\frac{m_a+m_b}{m_c}+1\right)^2-2\left(\frac{m_a+m_b}{m_c}\right)\right)+13=$$

$$=\sum\left(\left(\frac{m_a+m_c+m_b}{m_c}\right)^2-2\left(\frac{m_a+m_b}{m_c}\right)\right)+13=$$

$$=(m_a+m_b+m_c)^2\sum\frac{1}{m_c^2}-2\sum\left(\frac{m_a}{m_c}+\frac{m_b}{m_c}\right)+13\leq$$

$$\overset{Gotman\ II\ \&\ m_c\geq\sqrt{s(s-c)}}{\leq}\left(\frac{9R}{2}\right)^2\sum\frac{1}{s(s-c)}-2\sum\left(\frac{m_a}{m_c}+\frac{m_c}{m_a}\right)+13\leq$$

$$\overset{(2)\&\ AM-GM}{\leq}\frac{81R^2}{4}\cdot\frac{1}{3r^2}-2\sum 2+13=\frac{27}{4}\left(\frac{R}{r}\right)^2+1\ (3)$$

$$\sum\left(\frac{b+c}{a}\right)^2\overset{CBS}{\geq}\frac{1}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)^2\overset{AM-GM}{\geq}$$

$$\geq \frac{1}{3}\left(6\sqrt[6]{\frac{b}{a}\cdot\frac{c}{a}\cdot\frac{c}{b}\cdot\frac{a}{b}\cdot\frac{a}{c}\cdot\frac{b}{c}}\right)^2 = \frac{36}{3} = 12 \quad (4)$$

***We need to show*:**

$$\sum\left(\frac{b+c}{a}\right)^2 + \frac{R^4}{r^4} \geq \sum\left(\frac{m_a+m_b}{m_c}\right)^2 + 16$$

$$12 + \left(\frac{R}{r}\right)^4 \overset{using\ (4)\&\ (3)}{\geq} \frac{27}{4}\left(\frac{R}{r}\right)^2 + 1$$

$$x^4 - \frac{27}{4}x^2 + 11 \overset{\frac{R}{r}=x\geq 2\ (Euler)}{\geq} 0$$

$$4x^4 - 27x^2 + 44 \geq 0$$

$$(x-2)(4x^3 + 8x^2 - 11x - 22) \geq 0$$

$$(x-2)\left((4x^3 - 22) + x(8x - 11)\right) \geq 0$$

$$true\ as\ x \geq 2\ and\ (4x^3 - 11) > 0\ ,(8x - 11) > 0$$

Equality holds for an equilateral triangle.