ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$min\left(\sum_{cyc}\frac{1}{h_ah_b},\sum_{cyc}\frac{1}{r_ar_b}\right)\leq\frac{1}{3r^2}$$

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Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{1}{h_ah_b}=\sum_{cyc}\frac{ab}{4F^2}=\frac{1}{4F^2}\sum_{cyc}ab=\frac{1}{4r^2s^2}(s^2+r^2+4Rr)\leq\frac{1}{3r^2}\Leftrightarrow$$

$$\Leftrightarrow 3(s^2+r^2+4Rr)\leq 4s^2\Leftrightarrow 3r^2+12Rr\leq s^2 \text{ (to prove)}$$

$$s^2\overset{GERRETSEN}{\geq}16Rr-5r^2\geq 3r^2+12Rr\Leftrightarrow 4Rr\geq 8r^2\Leftrightarrow R\overset{EULER}{\geq}2r$$

$$\sum_{cyc}\frac{1}{r_ar_b}=\sum_{cyc}\frac{(s-a)(s-b)}{F^2}=\sum_{cyc}\frac{(s-a)(s-b)}{s(s-a)(s-b)(s-c)}=$$

$$=\sum_{cyc}\frac{1}{s(s-a)}=\frac{1}{s}\sum_{cyc}\frac{1}{s-a}=\frac{1}{s}\cdot\frac{4R+r}{rs}\leq\frac{1}{3r^2}\Leftrightarrow$$

$$\Leftrightarrow s^2\geq 12Rr+3r^2 \text{(to prove)}$$

$$s^2\overset{GERRETSEN}{\geq}16Rr-5r^2\geq 3r^2+12Rr\Leftrightarrow 4Rr\geq 8r^2\Leftrightarrow R\overset{EULER}{\geq}2r$$

Equality holds for $a=b=c$.