ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$R \geq \frac{1}{2}(acotA + bcotB + ccotC) \geq 2r$$

(Refinement for Euler's inequality)

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$$\sum_{cyc} acotA = \sum_{cyc} \frac{acosA}{sinA} = \sum_{cyc} \frac{2RsinAcosA}{sinA} = 2R\sum_{cyc} cosA =$$

$$= 2R\left(1 + \frac{r}{R}\right) = 2R + 2r \overset{EULER}{\overbrace{\geq}} 2 \cdot 2r + 2r = 6r$$

$$\sum_{cyc} acotA = \sum_{cyc} \frac{acosA}{sinA} = \sum_{cyc} \frac{2RsinAcosA}{sinA} = 2R\sum_{cyc} cosA \leq$$

$$\overset{JENSEN}{\overbrace{\leq}} 2R \cdot 3cos\left(\frac{A + B + C}{3}\right) = 6Rcos\frac{\pi}{3} = 3R$$

$$3R \geq \sum_{cyc} acotA \geq 6r$$

$$R \geq \frac{1}{2}\sum_{cyc} acotA \geq 2r$$

Equality holds for $a = b = c$.