ROMANIAN MATHEMATICAL MAGAZINE

Prove that in any ΔABC if $0 < C \leq A \leq \frac{\pi}{2}$ then:

$$\cos\frac{A-B}{2}\sin\frac{A}{2}\sin\frac{B}{2} \geq \frac{1}{4}$$

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Solution by Tapas Das-India

$$\text{We know that if } \Delta ABC \text{ acute triangle then:}$$
$$R + r \leq \max(h_a, h_b, h_c) \ \ (1)$$

$$\left(\begin{array}{c}\text{Reference: ERDOS' inequality:}\\ (\text{RMM famous inequalty marathon } 1-100\ (31) - www.ssmrmh.ro)\end{array}\right)$$

$$\text{As } C \leq B \leq A \text{ then } h_c \overset{(1)}{\geq} R + r \text{ and}$$

$$\frac{h_c}{r} = \frac{2s}{c} = 1 + \frac{a+b}{c} \geq \frac{R+r}{r} \ \text{ or,} \frac{a+b}{c} \geq \frac{R}{r} \ \ (\text{Bogdan Fuștei})\ (2)$$

$$\cos\frac{A-B}{2}\sin\frac{A}{2}\sin\frac{B}{2} \overset{Mollweide}{=} \frac{a+b}{c}\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2} = \frac{a+b}{c}\frac{r}{4R} \overset{(2)}{\geq} \frac{R}{r}\cdot\frac{r}{4R} = \frac{1}{4}$$

Equality holds for an equilateral triangle.