# ROMANIAN MATHEMATICAL MAGAZINE

**Prove that in any $\Delta$ ABC, if $\hat{A} \geq 90^\circ$ then :**

$$\sin A + \sin B + \sin C \leq 1 + \sqrt{2}$$

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$$\text{Let } f(x) = \sin x + 2\cos\frac{x}{2}\ \forall x \,\Big|\, \frac{\pi}{2} \leq x < \pi \text{ and then : } f'(x) = \cos x - \sin\frac{x}{2}$$

$$= 1 - 2\sin^2\frac{x}{2} - \sin\frac{x}{2} = \left(1 + \sin\frac{x}{2}\right)\left(1 - 2\sin\frac{x}{2}\right) < 0$$

$$\left(\because \frac{\pi}{4} \leq \frac{x}{2} < \frac{\pi}{2} \Rightarrow \sin\frac{x}{2} \geq \frac{1}{\sqrt{2}} > \frac{1}{2} \Rightarrow 1 - 2\sin\frac{x}{2} < 0\right) \Rightarrow f'(x) < 0$$

$$\Rightarrow f(x) \text{ is} \downarrow\ \forall x \,\Big|\, \frac{\pi}{2} \leq x < \pi \Rightarrow f(x) \leq f\left(\frac{\pi}{2}\right) = 1 + \frac{2}{\sqrt{2}}$$

$$\therefore \sin x + 2\cos\frac{x}{2} \leq 1 + \sqrt{2}\ \forall x \,\Big|\, \frac{\pi}{2} \leq x < \pi \rightarrow (1)$$

$$\text{Now, } \sin A + \sin B + \sin C = \sin A + 2\cos\frac{A}{2}\cos\frac{B-C}{2} \leq \sin A + 2\cos\frac{A}{2}$$

$$\left(\because 0 < \cos\frac{B-C}{2} \leq 1\right) \overset{\text{via } (1)}{\leq} 1 + \sqrt{2} \text{ and so, } \sin A + \sin B + \sin C \leq 1 + \sqrt{2}$$

$$\forall\, ABC \,\big|\, \hat{A} \geq 90^\circ, \text{"="} \text{ iff } \left(\hat{A} = 90^\circ, \hat{B} = \hat{C} = 45^\circ\right) \text{(QED)}$$