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In $\triangle ABC$ the following relationship holds:

$$\left(\csc \frac{A}{2}\right)^{\csc \frac{A}{2}} \cdot \left(\csc \frac{B}{2}\right)^{\csc \frac{B}{2}} \cdot \left(\csc \frac{C}{2}\right)^{\csc \frac{C}{2}} \geq 64$$

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Solution by Mirsadix Muzefferov-Azerbaijan

$$\begin{aligned} \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} &\leq \frac{3}{2} \\ \frac{1}{\sin \frac{A}{2}} + \frac{1}{\sin \frac{B}{2}} + \frac{1}{\sin \frac{C}{2}} &\geq \frac{9}{\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}} \geq \frac{9 \cdot 2}{3} = 6 \\ \left(\csc \frac{A}{2}\right)^{\csc \frac{A}{2}} \cdot \left(\csc \frac{B}{2}\right)^{\csc \frac{B}{2}} \cdot \left(\csc \frac{C}{2}\right)^{\csc \frac{C}{2}} &= \\ = \left(\csc \frac{A}{2}\right)^{\left(\csc \frac{A}{2}\right)} \cdot \left(\csc \frac{B}{2}\right)^{\left(\csc \frac{B}{2}\right)} \cdot \left(\csc \frac{C}{2}\right)^{\left(\csc \frac{C}{2}\right)} &\stackrel{\text{Weighted GM-HM}}{\geq} \\ \geq \left(\frac{\csc \frac{A}{2} + \csc \frac{B}{2} + \csc \frac{C}{2}}{\frac{\csc \frac{A}{2}}{\csc \frac{A}{2}} + \frac{\csc \frac{B}{2}}{\csc \frac{B}{2}} + \frac{\csc \frac{C}{2}}{\csc \frac{C}{2}}} \right)^{\csc \frac{A}{2} + \csc \frac{B}{2} + \csc \frac{C}{2}} &= \\ = \left(\frac{1}{3} \left(\frac{1}{\sin \frac{A}{2}} + \frac{1}{\sin \frac{B}{2}} + \frac{1}{\sin \frac{C}{2}} \right) \right)^{\sum_{\text{cyc}} \frac{1}{\sin \frac{A}{2}}} &\geq \left(\frac{6}{3} \right)^6 = 2^6 = 64 \end{aligned}$$

Equality holds for : $A = B = C$.