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In ΔABC the following relationship holds:

$$r_a sec\frac{A}{2}+r_b sec\frac{B}{2}+r_c sec\frac{C}{2}\ge 6\sqrt{3}r$$

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$$In\ \ \Delta ABC\ WLOG\ :\ a\le b\le c$$

$$Then\ r_a\le r_b\le r_c,\qquad \frac{1}{cos\frac{A}{2}}\le\frac{1}{cos\frac{B}{2}}\le\frac{1}{cos\frac{C}{2}}$$

$$\sum_{cyc}\frac{r_a}{cos\frac{A}{2}}\overset{Chebyshev}{\ge}\frac{1}{3}.\sum_{cyc}\frac{1}{cos\frac{A}{2}}.\sum_{cyc}r_a=$$

$$=\frac{1}{3}.(4R+r).\sum_{cyc}\frac{1}{cos\frac{A}{2}}\overset{AM-GM}{\ge}\frac{1}{3}.(4R+r).3.\sqrt[3]{\frac{1}{\prod_{cyc}cos\frac{A}{2}}}\ge$$

$$\ge(4R+r).\sqrt[3]{\frac{8}{3\sqrt{3}}}\overset{Euler}{\ge}9r.\frac{2}{\sqrt{3}}=6\sqrt{3}r$$

$$Equality\ \ holds\ for:\ \ a=b=c.$$