ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\cos A+\cos B+\cos C+\cos 2A+\cos 2B+\cos 2C\ge 0$$

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$$\cos A+\cos B+\cos C+\cos 2A+\cos 2B+\cos 2C$$

$$\sum\cos A+\sum\cos 2A=\sum\cos A+\sum(2\cos^2 A-1)=$$
$$=\sum\cos A+2\sum\cos^2 A=$$
$$=\sum\cos A+2\left(\sum\cos A\right)^2-4\sum\cos A\cos B-3=$$
$$=\left(1+\frac{r}{R}\right)+2\left(1+\frac{r}{R}\right)^2-4\frac{s^2+r^2-4R^2}{4R^2}-3\ge$$
$$\overset{Gerretsen}{\ge}\left(1+\frac{r}{R}\right)+2\left(1+\frac{r}{R}\right)^2-4\frac{4R^2+4Rr+3r^2+r^2-4R^2}{4R^2}-3=$$
$$=\left(1+\frac{r}{R}\right)+2\left(1+\frac{r}{R}\right)^2-\left(\frac{4r}{R}+\frac{4r^2}{R^2}\right)-3\overset{\frac{r}{R}=x\le\frac{1}{2}}{=}$$
$$=(1+x)+2(1+x)^2-(4x+4x^2)-3=$$

$$=(1+x)+2(1+2x+x^2)-(4x+4x^2)-3=-2x^2+x$$

$$\textit{We need to show according to the problem}:$$
$$-2x^2+x\ge 0\ or,x\le\frac{1}{2}\ true\ as\ \frac{r}{R}=x\le\frac{1}{2}(Euler)$$

Equality holds for an equilateral triangle.