ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}\ge\sqrt{\frac{3(4R+r)}{2R}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{r_b}+\frac{1}{r_c}=\frac{s-b+s-c}{F}=\frac{a}{F}\ or\ a=F\frac{r_b+r_c}{r_br_c}$$

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}=\sum\frac{r_a}{a}=\sum\frac{r_a}{F\frac{r_b+r_c}{r_br_c}}=\frac{r_ar_br_c}{F}\sum\frac{1}{r_b+r_c}=$$

$$=\frac{s^2r}{rs}\frac{3\sum r_ar_b+\sum r_a^2}{(\sum r_a)(\sum r_ar_b)-r_ar_br_c}=s.\frac{(4R+r)^2+s^2}{(4r+r)s^2-s^2r}=\frac{(4R+r)^2+s^2}{4Rs}$$

$$We\ need\ to\ show,:$$

$$\frac{(4R+r)^2+s^2}{4Rs}\ge\sqrt{\frac{3(4R+r)}{2R}}$$

$$\frac{((4R+r)^2+s^2)^2}{16R^2s^2}\ge\frac{3(4R+r)}{2R}$$

$$s^4+2s^2(4R+r)^2+(4R+r)^4\ge 24s^2R(4R+r)$$

$$s^4-2s^2(12R(4R+r)-(4R+r)^2)+(4R+r)^4\ge 0$$

$$s^4-2s^2(4R+r)(8R-r)+(4R+r)^4\ge 0$$

$$s^2\left(s^2-2(4R+r)(8R-r)\right)+(4R+r)^4\ge 0$$

$$s^2\big(3r(4R+r)-2(4R+r)(8R-r)\big)+(4R+r)^4\overset{s^2\ge 3r(4R+r)}{\ge} 0$$

$$(4R+r)s^2(3r-16R+2r)+(4R+r)^4\ge 0$$

$$(4R+r)^3-s^2(16R-5r)\ge 0$$

$$(4R+r)^3-(4R^2+4Rr+3r^2)(16R-5r)\overset{Gerretsen}{\ge} 0$$

$$(4x+1)^3-(4x^2+4x+3)(16x-5)\overset{\frac{R}{r}=x\ge 2}{\ge} 0$$

$$(64x^3+48x^2+12x+1)-(64x^3+44x^2+2x+5)\ge 0$$

$$x^2-4x+4\ge 0\ or,(x-2)^2\ge 0\ true$$

Equality holds for an equilateral triangle.