ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\left(\cos\frac{A}{2}\right)^{\cos\frac{B}{2}+\cos\frac{C}{2}}\left(\cos\frac{B}{2}\right)^{\cos\frac{C}{2}+\cos\frac{A}{2}}\left(\cos\frac{C}{2}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}}\le\left(\frac{\sqrt{3}}{2}\right)^{3\sqrt{3}}$$

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Solution by Tapas Das-India

$$Let:\ \cos\frac{A}{2}=x,\cos\frac{B}{2}=y,\cos\frac{C}{2}=z\ then:$$

$$x+y+z=\sum\cos\frac{A}{2}\overset{Jensen}{\le}3\ \cos\frac{\pi}{6}=\frac{3\sqrt{3}}{2}\ (1)$$

$$Given\ problem\ can\ be\ written\ as\ using\ above\ substitution:$$

$$\left(\cos\frac{A}{2}\right)^{\cos\frac{B}{2}+\cos\frac{C}{2}}\left(\cos\frac{B}{2}\right)^{\cos\frac{C}{2}+\cos\frac{A}{2}}\left(\cos\frac{C}{2}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}}\le\left(\frac{\sqrt{3}}{2}\right)^{3\sqrt{3}}$$

$$x^{y+z}.y^{z+x}.z^{x+y}\le\left(\frac{\sqrt{3}}{2}\right)^{3\sqrt{3}}$$

$$x^{y+z}.y^{z+x}.z^{x+y}\overset{AM-GM}{\le}\left(\frac{x(y+z)+y(z+x)+z(x+y)}{y+z+z+x+x+y}\right)^{x+y+y+z+z+x}=$$

$$=\left(\frac{xy+yz+zx}{x+y+z}\right)^{2(x+y+z)}\le\left(\frac{\frac{(x+y+z)^2}{3}}{x+y+z}\right)^{2(x+y+z)}=$$

$$=\left(\frac{x+y+z}{3}\right)^{2(x+y+z)}\overset{(1)}{\le}\left(\frac{3\sqrt{3}}{2}.\frac{1}{3}\right)^{2.\frac{3\sqrt{3}}{2}}=\left(\frac{\sqrt{3}}{2}\right)^{3\sqrt{3}}$$

Equality holds for an equilateral triangle.