ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{h_a}{l_a}+\frac{h_b}{l_b}+\frac{h_c}{l_c}\ge 1+\frac{4r}{R}$$

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$$\frac{h_a}{l_a}=\frac{\frac{2F}{a}}{\frac{2bc}{b+c}.\cos\frac{A}{2}}=\frac{F(b+c)}{abc.\cos\frac{A}{2}}=\frac{F(b+c)}{4FR.\cos\frac{A}{2}}=$$

$$\frac{2R(\sin B+\sin C)}{4R.\cos\frac{A}{2}}=\frac{2\sin\frac{B+C}{2}.\cos\frac{B-C}{2}}{2\cos\frac{A}{2}}=\frac{\cos\frac{A}{2}.\cos\frac{B-C}{2}}{\cos\frac{A}{2}}=\cos\frac{B-C}{2}$$

$$\overbrace{\cos\frac{B-C}{2}=\frac{b+c}{a}.\sin\frac{A}{2}}^{Molweide's}=\frac{b+c}{a}\sqrt{\frac{(p-b)(p-c)}{bc}}\overset{A-G}{\ge}$$

$$\ge\frac{2\sqrt{bc}}{a}\sqrt{\frac{(p-b)(p-c)}{bc}}=\frac{2}{a}\sqrt{(p-b)(p-c)}$$

$$\sum_{cyc}\frac{h_a}{l_a}\ge\sum_{cyc}\frac{2}{a}\sqrt{(p-b)(p-c)}\ge 3\left(\frac{8}{abc}(p-a)(p-b)(p-c)\right)^{\frac{1}{3}}=$$

$$=6\left(\frac{p.(p-a)(p-b)(p-c)}{abc.p}\right)^{\frac{1}{3}}=6\left(\frac{F^2}{4FR.p}\right)^{\frac{1}{3}}=6\left(\frac{F}{4R.p}\right)^{\frac{1}{3}}=$$

$$=6\left(\frac{p.r}{4R.p}\right)^{\frac{1}{3}}=6\left(\frac{2r}{8R}\right)^{\frac{1}{3}}=3\left(\frac{2r}{R}\right)^{\frac{1}{3}}$$

$$Let's\ \ prove\ \ that\ :\ 3\left(\frac{2r}{R}\right)^{\frac{1}{3}}\ge 1+\frac{4r}{R}$$

$$Let\ \frac{2r}{R}=x\,;\ \ Then\ \ 3\sqrt[3]{x}\ge 1+2x\to\ \ 27x\ge(1+2x)^3$$

$$8x^3+12x^2-21x+1\le 0$$

$$(x-1)(8x^2+20x-1)\le 0\ \to\ \ x\le 1\to\ \ R\ge 2r\ \ (Euler)$$