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In any Δ ABC the following relationship holds :

$$\prod_{cyc}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{\tan\frac{A}{2}+\tan\frac{B}{2}} \geq \left(\frac{2}{\sqrt{3}}\right)^{2\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{\tan\frac{A}{2}+\tan\frac{B}{2}} = \prod_{cyc}\left(\frac{4R}{s}\cos^2\frac{A}{2}\right)^{\frac{4R}{s}\cos^2\frac{A}{2}} \overset{\text{Weighted GM-HM}}{\geq}$$

$$\left(\frac{\frac{4R}{s}\sum_{cyc}\cos^2\frac{A}{2}}{\sum_{cyc}\frac{\frac{4R}{s}\cos^2\frac{A}{2}}{\frac{4R}{s}\cos^2\frac{A}{2}}}\right)^{\frac{4R}{s}\sum_{cyc}\cos^2\frac{A}{2}} = \left(\frac{\frac{4R}{s}\cdot\frac{4R+r}{2R}}{3}\right)^{\frac{4R}{s}\cdot\frac{4R+r}{2R}} = \left(\frac{2(4R+r)}{3s}\right)^{\frac{2(4R+r)}{s}} \overset{\text{Doucet or Trucht}}{\geq}$$

$$\left(\frac{2\sqrt{3}}{3}\right)^{2\sqrt{3}} \therefore \prod_{cyc}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^{\tan\frac{A}{2}+\tan\frac{B}{2}} \geq \left(\frac{2}{\sqrt{3}}\right)^{2\sqrt{3}} \ \forall\, \Delta\, ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)