ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{2s}{\sqrt[3]{4sRr}}$$

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Solution by Tapas Das-India

Lemma:

$$\frac{x+y+z}{\sqrt[3]{xyz}}\le\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

Proof:

$$\frac{x+y+z}{\sqrt[3]{xyz}}=\frac{x}{\sqrt[3]{xyz}}+\frac{y}{\sqrt[3]{xyz}}+\frac{z}{\sqrt[3]{xyz}}=$$

$$=\sqrt[3]{\frac{x^2}{yz}}+\sqrt[3]{\frac{y^2}{xz}}+\sqrt[3]{\frac{z^2}{yx}}=\sqrt[3]{\left(\frac{x}{y}\right)^2\cdot\left(\frac{y}{z}\right)}+\sqrt[3]{\left(\frac{y}{z}\right)^2\cdot\left(\frac{z}{x}\right)}+\sqrt[3]{\left(\frac{z}{x}\right)^2\cdot\left(\frac{x}{y}\right)}\overset{AM-GM}{\le}$$

$$\le\frac{\frac{x}{y}+\frac{x}{y}+\frac{y}{z}}{3}+\frac{\frac{y}{z}+\frac{y}{z}+\frac{z}{x}}{3}+\frac{\frac{z}{x}+\frac{z}{x}+\frac{x}{y}}{3}=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\overset{Lemma}{\ge}\frac{a+b+c}{\sqrt[3]{abc}}=\frac{2s}{\sqrt[3]{4Rrs}}$$

Equality holds for an equilateral triangle.