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In ΔABC the following relationship holds:

$$(\csc A)^{\cot\frac{A}{2}}(\csc B)^{\cot\frac{B}{2}}(\csc C)^{\cot\frac{C}{2}} \geq \left(\frac{2}{\sqrt{3}}\right)^{3\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum \cos A = 1 + \frac{r}{R} \overset{Euler}{\leq} \frac{3}{2} \ (1)$$

$$\sum \cot\frac{A}{2} = \frac{s}{r} \overset{Mitrinvic}{\geq} 3\sqrt{3} \ (2)$$

$$\sum \frac{\cot\frac{A}{2}}{\csc A} = \sum \sin A \ .\cot\frac{A}{2} = \sum 2\cos^2\frac{A}{2} = \sum (1 + \cos A) =$$
$$= 3 + \sum \cos A \overset{(1)}{\leq} 3 + \frac{3}{2} = \frac{9}{2} \ (3)$$

$$(\csc A)^{\cot\frac{A}{2}}(\csc B)^{\cot\frac{B}{2}}(\csc C)^{\cot\frac{C}{2}} \overset{Weighted\ GM\geq HM}{\geq}$$
$$\geq \left(\frac{\left(\sum \cot\frac{A}{2}\right)}{\sum \frac{\cot\frac{A}{2}}{\csc A}}\right)^{\sum \cot\frac{A}{2}} \overset{(2)\&(3)}{\geq} \left(\frac{3\sqrt{3}}{\frac{9}{2}}\right)^{3\sqrt{3}} = \left(\frac{2}{\sqrt{3}}\right)^{3\sqrt{3}}$$

$$Equality\ holds\ for\ A = B = C.$$