ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$\sum_{cyc} a\cot A < \frac{2}{3}(a+b+c)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a\cot A = \sum_{cyc}(2R\sin A\cot A) = 2R\sum_{cyc}\cos A = 2R\left(\frac{R+r}{R}\right)$$

$$= 2R+2r \overset{?}{<} \frac{2}{3}(a+b+c) \Leftrightarrow 2s \underbrace{\overset{?}{>}}_{①} 3R+3r \because \cos A\cos B\cos C = \frac{s^2-(2R+r)^2}{4R^2}$$

$$> 0 \text{ in an acute triangle} \therefore s > 2R+r \Rightarrow 2s > 4R+2r = 3R+3r+(R-2r)+r$$

$$\overset{\text{Euler}}{\geq} 3R+3r+r > 3R+3r \Rightarrow ① \text{ is true}$$

$$\therefore \sum_{cyc} a\cot A < \frac{2}{3}(a+b+c)\ \forall \text{ acute } \Delta \text{ ABC}$$