

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$2 \left(\sum_{\text{cyc}} \sin A \right) \left(\sum_{\text{cyc}} \cos \frac{A}{2} \right) \leq \frac{45}{4} + \left(\sum_{\text{cyc}} \cos A \right) \left(\sum_{\text{cyc}} \sin \frac{A}{2} \right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & \frac{45}{4} + \left(\sum_{\text{cyc}} \cos A \right) \left(\sum_{\text{cyc}} \sin \frac{A}{2} \right) - 2 \left(\sum_{\text{cyc}} \sin A \right) \left(\sum_{\text{cyc}} \cos \frac{A}{2} \right) \stackrel{\text{CBS}}{\geq} \\ & \quad \quad \quad 0 < \cos \frac{B-C}{2} \leq 1 \text{ and analogs} \\ & \frac{45}{4} + \left(\frac{R+r}{R} \right) \left(\sum_{\text{cyc}} \frac{2 \cos \frac{B+C}{2} \cos \frac{B-C}{2}}{2 \cos \frac{B-C}{2}} \right) - 2 \left(\frac{S}{R} \right) \cdot \sqrt{3 \sum_{\text{cyc}} \cos^2 \frac{A}{2}} \stackrel{\text{Gerretsen}}{\geq} \\ & \frac{45}{4} + \left(\frac{R+r}{R} \right) \left(\sum_{\text{cyc}} \frac{\cos B + \cos C}{2} \right) - \frac{2 \cdot \sqrt{4R^2 + 4Rr + 3r^2}}{R} \cdot \sqrt{\frac{3(4R+r)}{2R}} \stackrel{?}{\geq} 0 \\ & \Leftrightarrow \frac{45}{4} + \left(\frac{R+r}{R} \right)^2 \stackrel{?}{\geq} \frac{2 \cdot \sqrt{4R^2 + 4Rr + 3r^2}}{R} \cdot \sqrt{\frac{3(4R+r)}{2R}} \\ & \Leftrightarrow \left(\frac{45R^2 + 4(R+r)^2}{4R^2} \right)^2 \stackrel{?}{\geq} \frac{6(4R+r)(4R^2 + 4Rr + 3r^2)}{R^3} \\ & \Leftrightarrow (45R^2 + 4(R+r)^2)^2 - 96R(4R+r)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\geq} 0 \\ & \Leftrightarrow 865t^4 - 1136t^3 - 1080t^2 - 224t + 16 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \\ & \Leftrightarrow (t-2)(865t^3 + 594t^2 + 108t - 8) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \\ & \therefore 2 \left(\sum_{\text{cyc}} \sin A \right) \left(\sum_{\text{cyc}} \cos \frac{A}{2} \right) \leq \frac{45}{4} + \left(\sum_{\text{cyc}} \cos A \right) \left(\sum_{\text{cyc}} \sin \frac{A}{2} \right) \forall \Delta ABC, \end{aligned}$$

" = " iff ΔABC is equilateral (QED)