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In acute ΔABC the following relationship holds:

$$(1+\cos A\cos B)^{\cos C}(1+\cos B\cos C)^{\cos A}(1+\cos C\cos A)^{\cos B}\le\frac{5\sqrt{5}}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\cos A=1+\frac{r}{R}\overset{Euler}{\le}\frac{3}{2}\ (1)$$

$$(1+\cos A\cos B)^{\cos C}(1+\cos B\cos C)^{\cos A}(1+\cos C\cos A)^{\cos B}$$

$$\overset{AM-GM}{\le}\left(\frac{\sum\cos C\,(1+\cos A\cos B)}{\sum\cos A}\right)^{\sum\cos A}=$$

$$=\left(\frac{\sum\cos A+3\cos A\cos B\cos C}{\sum\cos A}\right)^{\sum\cos A}\overset{AM-GM}{\le}$$

$$\le\left(\frac{\sum\cos A+\frac{3(\sum\cos A)^3}{27}}{\sum\cos A}\right)^{\sum\cos A}=\left(1+\frac{1}{9}\left(\sum\cos A\right)^2\right)^{\sum\cos A}\overset{(1)}{\le}$$

$$\le\left(1+\frac{1}{9}\left(\frac{3}{2}\right)^2\right)^{\frac{3}{2}}=\left(\frac{5}{4}\right)^{\frac{3}{2}}=\frac{5\sqrt{5}}{8}$$

Equality holds for an equilateral triangle.