ROMANIAN MATHEMATICAL MAGAZINE

In acute ΔABC the following relationship holds:

$$(\cos A+\cos B)^{\cos C}(\cos B+\cos C)^{\cos A}(\cos C+\cos A)^{\cos B}\le 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\cos A+\cos B+\cos C=1+\frac{r}{R}\overset{Euler}{\le}\frac{3}{2}\ (1)$$

$$Using\ \ weighted\ AM\ge GM:$$

$$(\cos A+\cos B)^{\cos C}(\cos B+\cos C)^{\cos A}(\cos C+\cos A)^{\cos B}\le$$

$$\le\left(\frac{\sum\cos A\,(\cos B+\cos C)}{\cos A+\cos B+\cos C}\right)^{\cos A+\cos B+\cos C}=$$

$$=\left(\frac{2\sum\cos B\cos A}{\cos A+\cos B+\cos C}\right)^{\cos A+\cos B+\cos C}\le\left(\frac{\frac{2(\cos A+\cos B+\cos C)^2}{3}}{\cos A+\cos B+\cos C}\right)^{\cos A+\cos B+\cos C}=$$

$$=\left(\frac{2}{3}\,(\cos+\cos+\cos\right)^{cos\,A+cos\,B+cos\,C}\overset{(1)}{\le}\left(\frac{2}{3}\frac{3}{2}\right)^{\frac{3}{2}}=(1)^{\frac{3}{2}}=1$$

Equality holds for an equilateral triangle.