ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{b+c-a}{c+a-b}+\frac{c+a-b}{a+b-c}+\frac{a+b-c}{b+c-a}\ge\sqrt[3]{\frac{s^2}{r^2}}$$

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Solution by Tapas Das-India

$\boldsymbol{Lemma}$:

$$\forall\, x,y,z>0\,,\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{x+y+z}{\sqrt[3]{xyz}}$$

$\boldsymbol{Proof}$:

$$\frac{x+y+z}{\sqrt[3]{xyz}}=\frac{x}{\sqrt[3]{xyz}}+\frac{y}{\sqrt[3]{xyz}}+\frac{z}{\sqrt[3]{xyz}}$$

$$=\sqrt[3]{\frac{x^2}{yz}}+\sqrt[3]{\frac{y^2}{xz}}+\sqrt[3]{\frac{z^2}{yx}}=\sqrt[3]{\left(\frac{x}{y}\right)^2\cdot\left(\frac{y}{z}\right)}+\sqrt[3]{\left(\frac{y}{z}\right)^2\cdot\left(\frac{z}{x}\right)}+\sqrt[3]{\left(\frac{z}{x}\right)^2\cdot\left(\frac{x}{y}\right)}\overset{AM-GM}{\le}$$

$$\le\frac{\frac{x}{y}+\frac{x}{y}+\frac{y}{z}}{3}+\frac{\frac{y}{z}+\frac{y}{z}+\frac{z}{x}}{3}+\frac{\frac{z}{x}+\frac{z}{x}+\frac{x}{y}}{3}=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

$$\frac{b+c-a}{c+a-b}+\frac{c+a-b}{a+b-c}+\frac{a+b-c}{b+c-a}=\frac{s-a}{s-b}+\frac{s-b}{s-c}+\frac{s-c}{s-a}\ge$$

$$\overset{Lemma}{\ge}\frac{s-a+s-b+s-c}{\sqrt[3]{(s-a)(s-b)(s-c)}}=\frac{s}{\sqrt[3]{sr^2}}=\sqrt[3]{\frac{s^3}{sr^2}}=\sqrt[3]{\frac{s^2}{r^2}}$$

Equality holds for an equilateral triangle.