ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\ge\frac{(4R+r)}{\sqrt[3]{s^2r}}$$

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Lemma:

$$\forall\, x,y,z>0\,,\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\ge\frac{x+y+z}{\sqrt[3]{xyz}}$$

Proof:

$$\frac{x+y+z}{\sqrt[3]{xyz}}=\frac{x}{\sqrt[3]{xyz}}+\frac{y}{\sqrt[3]{xyz}}+\frac{z}{\sqrt[3]{xyz}}=$$

$$=\sqrt[3]{\frac{x^2}{yz}}+\sqrt[3]{\frac{y^2}{xz}}+\sqrt[3]{\frac{z^2}{yx}}=\sqrt[3]{\left(\frac{x}{y}\right)^2\cdot\left(\frac{y}{z}\right)}+\sqrt[3]{\left(\frac{y}{z}\right)^2\cdot\left(\frac{z}{x}\right)}+\sqrt[3]{\left(\frac{z}{x}\right)^2\cdot\left(\frac{x}{y}\right)}\overset{AM-GM}{\le}$$

$$\le\frac{\frac{x}{y}+\frac{x}{y}+\frac{y}{z}}{3}+\frac{\frac{y}{z}+\frac{y}{z}+\frac{z}{x}}{3}+\frac{\frac{z}{x}+\frac{z}{x}+\frac{x}{y}}{3}=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}+\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}+\frac{\tan\frac{C}{2}}{\tan\frac{A}{2}}\overset{Lemma}{\ge}\frac{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}{\sqrt[3]{\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}}=\frac{\frac{4R+r}{s}}{\sqrt[3]{\frac{r}{s}}}=\frac{4R+r}{\sqrt[3]{s^3\cdot\frac{r}{s}}}=\frac{(4R+r)}{\sqrt[3]{s^2r}}$$

Equality holds for an equilateral triangle.