ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}}\geq(2R-r)\sqrt[3]{\frac{2}{Rr^2}}$$

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Lemma:

$$\forall\, x,y,z>0\,,\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\frac{x+y+z}{\sqrt[3]{xyz}}$$

Proof:

$$\frac{x+y+z}{\sqrt[3]{xyz}}=\frac{x}{\sqrt[3]{xyz}}+\frac{y}{\sqrt[3]{xyz}}+\frac{z}{\sqrt[3]{xyz}}=$$

$$=\sqrt[3]{\frac{x^2}{yz}}+\sqrt[3]{\frac{y^2}{xz}}+\sqrt[3]{\frac{z^2}{yx}}=\sqrt[3]{\left(\frac{x}{y}\right)^2\cdot\left(\frac{y}{z}\right)}+\sqrt[3]{\left(\frac{y}{z}\right)^2\cdot\left(\frac{z}{x}\right)}+\sqrt[3]{\left(\frac{z}{x}\right)^2\cdot\left(\frac{x}{y}\right)}\overset{AM-GM}{\leq}$$

$$\leq\frac{\frac{x}{y}+\frac{x}{y}+\frac{y}{z}}{3}+\frac{\frac{y}{z}+\frac{y}{z}+\frac{z}{x}}{3}+\frac{\frac{z}{x}+\frac{z}{x}+\frac{x}{y}}{3}=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

$$\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}+\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}}\overset{lemma}{\geq}\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}}{\sqrt[3]{\sin^2\frac{A}{2}\sin^2\frac{A}{2}\sin^2\frac{A}{2}}}=\left(1-\frac{r}{2R}\right)\frac{1}{\sqrt[3]{\left(\frac{r}{4R}\right)^2}}=$$

$$=\frac{2R-r}{2R}\sqrt[3]{\frac{16R^2}{r^2}}=(2R-r)\sqrt[3]{\frac{16R^2}{8R^3r^2}}=(2\,R-r)\sqrt[3]{\frac{2}{Rr^2}}$$

Equality holds for an equilateral triangle.