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In ΔABC the following relationship holds:

$$\frac{a^2}{\cos^2\frac{A}{2}}+\frac{b^2}{\cos^2\frac{B}{2}}+\frac{c^2}{\cos^2\frac{C}{2}}\geq 12R^2$$

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Lemma: $\sum\sin^2\frac{A}{2}\geq\frac{3}{4}$

Proof: *Let* $f(x)=\sin^2\left(\frac{x}{2}\right)$, $f'(x)=\frac{\sin(x)}{2}$, $f''(x)=\frac{\cos(x)}{2}$ *and* $x\in\left(0,\ \frac{\pi}{2}\right)$

Since $f''(x)>0$ *for all* $x\in\left(0,\frac{\pi}{2}\right)$, *the function* f *is convex on this interval, and therefore Jensen's inequality can be applied*:

$$\sum\sin^2\left(\frac{A}{2}\right)\geq 3\cdot\sin^2\left(\frac{A+B+C}{6}\right)=3\cdot\sin^2\left(\frac{\pi}{6}\right)=\frac{3}{4}$$

$$\frac{a^2}{\cos^2\frac{A}{2}}+\frac{b^2}{\cos^2\frac{B}{2}}+\frac{c^2}{\cos^2\frac{C}{2}}=\sum\frac{a^2}{\cos^2\left(\frac{A}{2}\right)}=\sum\frac{\left(2R\cdot 2\cdot\sin\left(\frac{A}{2}\right)\cdot\cos\left(\frac{A}{2}\right)\right)^2}{\cos^2\left(\frac{A}{2}\right)}=$$

$$=16R^2\cdot\sum\sin^2\left(\frac{A}{2}\right)\geq 16R^2\cdot\frac{3}{4}=12R^2$$

Equality holds for: $a=b=c$.