ROMANIAN MATHEMATICAL MAGAZINE

Inspired by a problem of Professor Daniel Sitaru

In any acute Δ ABC the following relationship holds :

$$8m_am_bm_c \ge (w_a+w_b)(w_b+w_c)(w_c+w_a) \ge 8s_as_bs_c$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{w_a}\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{1}{h_a}}\cdot\sqrt{\sum_{cyc}\frac{h_a}{w_a^2}}=\sqrt{\frac{1}{2Rr}\cdot\sum_{cyc}\frac{bc}{bc-\frac{a^2bc}{(b+c)^2}}}$$

$$=\sqrt{\frac{1}{2Rr}\cdot\sum_{cyc}\frac{(s+s-a)^2}{4s(s-a)}}=\sqrt{\frac{1}{2Rr}\cdot\left(\frac{s(4Rr+r^2)}{4r^2s}+\frac{s}{4s}+\frac{3}{2}\right)}=\sqrt{\frac{R+2r}{2Rr^2}}\overset{?}{\le}\frac{3R+2r}{4Rr}$$

$$\Leftrightarrow (R-2r)^2\overset{?}{\ge}0\to \text{true}\therefore \sum_{cyc}\frac{1}{w_a}\le\frac{3R+2r}{4Rr}\to (1)\ \textit{and now},\ 8\prod_{cyc} m_a=$$

$$8\prod_{cyc} w_a\cdot\prod_{cyc}\frac{m_a}{w_a}\overset{Lascu}{\ge}8\prod_{cyc} w_a\cdot\prod_{cyc}\frac{\frac{b+c}{2}\cos\frac{A}{2}}{\frac{2bc\cos\frac{A}{2}}{b+c}}=8\prod_{cyc} w_a\cdot\prod_{cyc}\frac{(b+c)^2}{4bc}$$

$$=8\prod_{cyc} w_a\cdot\frac{4s^2(s^2+2Rr+r^2)^2}{64.16R^2r^2s^2}\therefore 8m_am_bm_c\ge\prod_{cyc} w_a\cdot\frac{(s^2+2Rr+r^2)^2}{32R^2r^2}\to (2)$$

$$\textit{and again},\ \prod_{cyc}(w_b+w_c)=\prod_{cyc} w_a\cdot\left(\left(\sum_{cyc} w_a\right)\left(\sum_{cyc}\frac{1}{w_a}\right)-1\right)$$

$$\overset{via\ (1)}{\le}\prod_{cyc} w_a\cdot\left(\sqrt{\frac{2s^2+41Rr+26r^2}{2}}\cdot\left(\frac{3R+2r}{4Rr}\right)-1\right)$$

(Reference : Inequality in Triangle by Dang Ngoc Minh – 113; published at www.ssmrmh.ro)

$$\overset{via\ (2)}{\Rightarrow}\ \text{it suffices to prove}:\ \frac{(s^2+2Rr+r^2)^2}{32R^2r^2}+1\overset{?}{\ge}\sqrt{\frac{2s^2+41Rr+26r^2}{2}}\cdot\left(\frac{3R+2r}{4Rr}\right)$$

$$\Leftrightarrow\frac{((s^2+2Rr+r^2)^2+32R^2r^2)^2}{32R^2r^2}\overset{?}{\ge}(2s^2+41Rr+26r^2)(3R+2r)^2$$

$$\Leftrightarrow s^8+(8Rr+4r^2)s^6+r^2(88R^2+24Rr+6r^2)s^4-$$
$$r^2(576R^4+480R^3r+80R^2r^2-24Rr^3-4r^4)s^2-$$
$$r^3(11808R^5+21936R^4r+14944R^3r^2+3240R^2r^3-8Rr^4-r^5)\underset{(*)}{\overset{?}{\ge}}0\ \textit{and}\ \because P=$$

$$(s^2-2R^2-8Rr-3r^2)^4+8(R^2+5Rr+2r^2)(s^2-2R^2-8Rr-3r^2)^3$$
$$\overset{Walker}{\ge}0\therefore \text{to prove } (*), \text{it suffices to prove : LHS of } (*)\overset{?}{\ge}P$$

$$\Leftrightarrow (3R^4+30R^3r+95R^2r^2+60Rr^3+12r^4)s^4 - (8R^6+108R^5r+594R^4r^2+1136R^3r^3+898R^2r^4+312Rr^5+40r^6)s^2 + 6R^8+104R^7r+712R^6r^2+960R^5r^3+1613R^4r^4+2114R^3r^5+1503R^2r^6+460Rr^7+44r^8 \overset{?}{\underset{(**)}{\geq}} 0 \text{ and } \because Q =$$

$$(3R^4+30R^3r+95R^2r^2+60Rr^3+12r^4)(s^2-2R^2-8Rr-3r^2)^2 \overset{\text{Walker}}{\geq} 0$$

$$\therefore \text{ to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\geq} Q \Leftrightarrow$$

$$(2R^6+30R^5r+142R^4r^2+402R^3r^3+340R^2r^4+120Rr^5+16r^6)s^2 \overset{?}{\underset{(***)}{\geq}}$$

$$3R^8+56R^7r+428R^6r^2+2372R^5r^3+4521R^4r^4+3830R^3r^5+1572R^2r^6+328Rr^7+32r^8 \text{ and finally, LHS of } (***) \overset{\text{Walker}}{\geq}$$

$$(2R^6+30R^5r+142R^4r^2+402R^3r^3+340R^2r^4+120Rr^5+16r^6)(2R^2+8Rr+3r^2)$$

$$\overset{?}{\geq} \text{RHS of } (***) \Leftrightarrow$$

$$t^8+20t^7+102t^6-342t^5-199t^4+336t^3+440t^2+160t+16 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow$$

$$(t-2)\left((t-2)(t^6+24t^5+194t^4+338t^3+377t^2+492t+900)+1792\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore 8m_am_bm_c \geq \prod_{\text{cyc}}(w_b+w_c)$$

$$\forall \text{ acute } \Delta\, ABC \text{ and } \forall \text{ acute } \Delta\, ABC, \frac{m_a}{w_a} \overset{\text{Tsintsifas}}{\leq} \frac{b^2+c^2}{2bc} \text{ and analogs} \Rightarrow w_a \geq s_a$$

$$\text{and analogs} \therefore (w_a+w_b)(w_b+w_c)(w_c+w_a) \overset{\text{Cesaro}}{\geq} 8w_aw_bw_c \geq 8s_as_bs_c$$

$$\therefore 8m_am_bm_c \geq (w_a+w_b)(w_b+w_c)(w_c+w_a) \geq 8s_as_bs_c \ \forall \text{ acute } \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$