In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{r}+3-\sqrt{2}} \le \frac{\sqrt{r_b r_c}}{h_a}+\frac{\sqrt{r_c r_a}}{h_b}+\frac{\sqrt{r_a r_b}}{h_c} \le \sqrt{\frac{2R}{r}+1}$$

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$$\sum_{cyc}\frac{s(s-a)}{h_a^2}=\frac{s}{4r^2s^2}\sum_{cyc}a^2(s-a)$$

$$=\frac{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{4r^2s}=\frac{4Rrs-8r^2s}{4r^2s}\Rightarrow \sum_{cyc}\frac{r_b r_c}{h_a^2}\overset{①}{=}\frac{R}{r}+1 \text{ and}$$

$$\text{now, }\sum_{cyc}\left(\frac{\sqrt{r_b r_c}}{h_a}.\frac{\sqrt{r_c r_a}}{h_b}\right)=R.\frac{\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}}{2r^2s^2}.\sum_{cyc}\frac{h_a}{\sqrt{s(s-a)}}$$

$$=\frac{Rs.rs.2rs}{2r^2s^2.\sqrt{s}}.\sum_{cyc}\frac{1}{a.\sqrt{s-a}}=\frac{Rs}{\sqrt{s}}.\sum_{cyc}\frac{1}{a.\sqrt{s-a}}\overset{\text{Bergstrom}}{\ge}\frac{Rs}{\sqrt{s}}.\frac{9}{\sum_{cyc}(a.\sqrt{s-a})}$$

$$=\frac{Rs}{\sqrt{s}}.\frac{9}{\sum_{cyc}\sqrt{a}.\sqrt{a(s-a)}}\overset{\text{CBS}}{\ge}\frac{Rs}{\sqrt{s}}.\frac{9}{\sqrt{2s}.\sqrt{\sum_{cyc}a(s-a)}}$$

$$=\frac{Rs}{\sqrt{s}}.\frac{9}{\sqrt{2s}.\sqrt{s(2s)-2(s^2-4Rr-r^2)}}=\frac{Rs}{2s}.\frac{9}{\sqrt{4Rr+r^2}}\overset{\text{Euler}}{\ge}\frac{9R}{2.\sqrt{\frac{9Rr}{2}}}$$

$$\therefore 2\sum_{cyc}\left(\frac{\sqrt{r_b r_c}}{h_a}.\frac{\sqrt{r_c r_a}}{h_b}\right)\ge 3\sqrt{2}.\sqrt{\frac{R}{r}}\overset{\text{via }①}{\Rightarrow}\sum_{cyc}\frac{r_b r_c}{h_a^2}+2\sum_{cyc}\left(\frac{\sqrt{r_b r_c}}{h_a}.\frac{\sqrt{r_c r_a}}{h_b}\right)\ge$$

$$\frac{R}{r}+1+3\sqrt{2}.\sqrt{\frac{R}{r}}\overset{?}{\ge}\left(\sqrt{\frac{R}{r}}+3-\sqrt{2}\right)^2=\frac{R}{r}+11-6\sqrt{2}+2.\sqrt{\frac{R}{r}}.(3-\sqrt{2})$$

$$\Leftrightarrow \sqrt{\frac{R}{r}}.(5\sqrt{2}-6)\overset{?}{\ge}10-6\sqrt{2}\rightarrow \text{true}\because \sqrt{\frac{R}{r}}.(5\sqrt{2}-6)\overset{\text{Euler}}{\ge}\sqrt{2}.(5\sqrt{2}-6)$$

$$(\because 5\sqrt{2}-6>0)=10-6\sqrt{2}\Rightarrow\left(\sum_{cyc}\frac{\sqrt{r_b r_c}}{h_a}\right)^2\ge\left(\sqrt{\frac{R}{r}+3-\sqrt{2}}\right)^2$$

$$\Rightarrow \boxed{\sum_{cyc}\frac{\sqrt{r_b r_c}}{h_a}\ge\sqrt{\frac{R}{r}+3-\sqrt{2}}}\text{ and again, }\sum_{cyc}\frac{\sqrt{r_b r_c}}{h_a}=\sum_{cyc}\frac{\sqrt{s(s-a)}.a}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

$$=\sum_{cyc}\frac{y+z}{2\sqrt{yz}}\ (x=s-a, y=s-b, z=s-c)=$$

$$\sum_{cyc}\frac{\beta^2+\gamma^2}{2\beta\gamma}\ (\alpha=\sqrt{x},\beta=\sqrt{y},\gamma=\sqrt{z})=\frac{1}{2\alpha\beta\gamma}\left(\left(\sum_{cyc}\alpha\right)\left(\sum_{cyc}\alpha\beta\right)-3\alpha\beta\gamma\right)$$

$$\Rightarrow\sum_{cyc}\frac{\sqrt{r_b r_c}}{h_a}-1=\frac{1}{2\alpha\beta\gamma}\left(\left(\sum_{cyc}\alpha\right)\left(\sum_{cyc}\alpha\beta\right)-5\alpha\beta\gamma\right)\overset{?}{\le}\sqrt{\frac{2R}{r}}$$

$$=\sqrt{\frac{(y+z)(z+x)(x+y)}{2xyz}}=\sqrt{\frac{(\alpha^2+\beta^2)(\beta^2+\gamma^2)(\gamma^2+\alpha^2)}{2\alpha^2\beta^2\gamma^2}}\Leftrightarrow$$

$$2\left(\left(\sum_{cyc}\alpha^2\right)\left(\sum_{cyc}\alpha^2\beta^2\right)-\alpha^2\beta^2\gamma^2\right)\boxed{\overset{?}{\underset{(*)}{\ge}}}\left(\left(\sum_{cyc}\alpha\right)\left(\sum_{cyc}\alpha\beta\right)-5\alpha\beta\gamma\right)^2$$

Since $\alpha,\beta,\gamma>0\ \therefore$ assigning $\beta+\gamma=X,\gamma+\alpha=Y,\alpha+\beta=Z\Rightarrow X+Y-Z=2\gamma>0,Y+Z-X=2\alpha>0,Z+X-Y=2\beta>0\Rightarrow X+Y>Z,Y+Z>X,Z+X>Y$ $\Rightarrow$ X, Y, Z form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s_0,R_0,r_0$ (say); then : $\sum_{cyc}\alpha=s_0,\alpha\beta\gamma=r_0^2s_0,\sum_{cyc}\alpha\beta=4R_0r_0+r_0^2$,

$\sum_{cyc}a^2=s_0^2-8R_0r_0-2r_0^2,\sum_{cyc}\alpha^2\beta^2=r_0^2\left((4R_0+r_0)^2-2s_0^2\right)$, and then,

$$(*)\text{ becomes}: 2\left(\left(s_0^2-8R_0r_0-2r_0^2\right)\left(r_0^2\left((4R_0+r_0)^2-2s_0^2\right)\right)-r_0^4s_0^2\right)\overset{?}{\ge}\left((s_0)(4R_0r_0+r_0^2)-5r_0^2s_0\right)^2\Leftrightarrow$$

$$4r_0^2\left(-s_0^4+(4R_0^2+20R_0r_0-2r_0^2)s_0^2-r_0(4R_0+r_0)^3\right)\overset{?}{\ge}0$$

Indeed, Rouche $\Rightarrow s_0^2-(m-n)\ge0$ and $s_0^2-(m+n)\le0$, where $m=2R_0^2+10R_0r_0-r_0^2$ and $n=2(R_0-2r_0).\sqrt{R_0^2-2R_0r_0}$

$$\therefore\left(s_0^2-(m+n)\right)\left(s_0^2-(m-n)\right)\le0\Rightarrow s_0^4-(4R_0^2+20R_0r_0-2r_0^2)s_0^2+r_0(4R_0+r_0)^3\le0\Rightarrow(*)\text{ is true}\ \therefore\sum_{cyc}\frac{\sqrt{r_br_c}}{h_a}\le\sqrt{\frac{2R}{r}}+1$$

$$\therefore\sqrt{\frac{R}{r}}+3-\sqrt{2}\le\sum_{cyc}\frac{\sqrt{r_br_c}}{h_a}\le\sqrt{\frac{2R}{r}}+1\ \forall\,\Delta\,ABC,''=''\text{ iff }a=b=c\ (QED)$$