ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$m_a + m_b + m_c \le g_a + g_b + g_c + 2(R - 2r)$$

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$$\sum_{cyc} g_a g_b \ge 4Rr + r^2 + \frac{r(s^2 + 4Rr + r^2)}{R}$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 70; Relation (m); published at www.ssmrmh.ro)

$$\therefore \left(g_a + g_b + g_c + 2(R - 2r)\right)^2 =$$

$$\sum_{cyc} g_a^2 + 2\sum_{cyc} g_a g_b + 4(R - 2r)^2 + 4(R - 2r)\left(\sum_{cyc} g_a\right) \overset{\text{Bogdan Fustei}}{=}$$

$$\sum_{cyc}(s - a)^2 + 2r\sum_{cyc} h_a + 2\sum_{cyc} g_a g_b + 4(R - 2r)^2 + 4(R - 2r)\left(\sum_{cyc} g_a\right) \ge$$

$$s^2 - 8Rr - 2r^2 + \frac{r(s^2 + 4Rr + r^2)}{R} + 2\left(4Rr + r^2 + \frac{r(s^2 + 4Rr + r^2)}{R}\right) +$$

$$4(R - 2r)^2 + 4(R - 2r)\left(\frac{s^2 + 4Rr + r^2}{2R}\right) \overset{?}{\ge} 4s^2 - 16Rr + 5r^2$$

$$\Leftrightarrow 4R^3 + 8R^2r + 9Rr^2 - r^3 \underbrace{\overset{?}{\ge}}_{(*)} (R + r)s^2 \text{ and indeed, } (R + r)s^2 \overset{\text{Gerretsen}}{\le}$$

$$(R + r)(4R^2 + 4Rr + 3r^2) \overset{?}{\le} 4R^3 + 8R^2r + 9Rr^2 - r^3 \Leftrightarrow 2r^2(R - 2r) \overset{?}{\ge} 0$$

$$\rightarrow \text{true via Euler} \Rightarrow (*) \text{ is true} \therefore g_a + g_b + g_c + 2(R - 2r) \ge \sqrt{4s^2 - 16Rr + 5r^2}$$

$$\overset{\text{Chu-Yang}}{\ge} m_a + m_b + m_c \Rightarrow m_a + m_b + m_c \le g_a + g_b + g_c + 2(R - 2r)\ \forall\ \Delta\ ABC,$$

$$'' = '' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$