# ROMANIAN MATHEMATICAL MAGAZINE

**In any Δ ABC the following relationship holds :**

$$\frac{p_a - m_a}{a} + \frac{p_b - m_b}{b} + \frac{p_c - m_c}{c} < \frac{1}{3}$$

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$$p_a - m_a \le \frac{a|b-c|}{2(2s+a)} \text{ and analogs}$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 56;\\ \text{published at www.ssmrmh.ro}\end{array}\right)$$

$$\therefore \sum_{cyc} \frac{p_a - m_a}{a} < \sum_{cyc} \frac{|b-c|}{2(2s+a)} \Rightarrow \left(\sum_{cyc} \frac{p_a - m_a}{a}\right)^2 <$$

$$\frac{1}{4}\left(\sum_{cyc} \frac{(b-c)^2}{(2s+a)^2} + 2\sum_{cyc} \frac{|b-c||c-a|}{(2s+a)(2s+b)}\right) <$$

$$\frac{1}{4}\cdot\left(\begin{array}{c}\frac{1}{4s^2(9s^2+6Rr+r^2)^2}\cdot\sum_{cyc}\left((b-c)^2(2s+b)^2(2s+c)^2\right) + \\ \frac{2}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc}\left(ab(2s+c)\right)\end{array}\right)$$

$$\therefore \left(\sum_{cyc} \frac{p_a - m_a}{a}\right)^2 \overset{①}{<} \frac{1}{4}\left(\frac{1}{4s^2(9s^2+6Rr+r^2)^2}\cdot\left(\begin{array}{c}\sum_{cyc}\left((b-c)^2(2s+b)^2(2s+c)^2\right) + \\ 8s^2(9s^2+6Rr+r^2)(s^2+10Rr+r^2)\end{array}\right)\right)$$

$$\text{Now, } \sum_{cyc}\left((b-c)^2(2s+b)^2(2s+c)^2\right) = \sum_{cyc}\left((b-c)^2(8s^2-2sa+bc)^2\right)$$

$$= \sum_{cyc}\left((b-c)^2(64s^4+16bcs^2+b^2c^2-32s^3a-4abcs+4a^2s^2)\right)$$

$$= 128s^4(s^2-12Rr-3r^2) +$$
$$16s^2\left(2(s^4-(4Rr+r^2)^2) - 8Rrs^2 - 2((s^2+4Rr+r^2)^2-16Rrs^2)\right) +$$
$$2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2) - 48R^2r^2s^2 -$$
$$2\left((s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2)\right) - 64s^4(s^2-14Rr+r^2) -$$
$$32Rrs^2(s^2-12Rr-3r^2) + 8s^2((s^2+4Rr+r^2)^2-16Rrs^2) - 64Rrs^4$$

$$\left(\because (b-c)^2 = \sum_{cyc} a^2 - a^2 - 2bc\right)$$

$$= 4(18s^6 - (168Rr+125r^2)s^4 - r^2(116R^2+84Rr+16r^2) - r^3(4R+r)^3)$$

$$\therefore \text{ via } ①, \left(\sum_{cyc} \frac{p_a - m_a}{a}\right)^2 <$$

$$\frac{4(18s^6 - (168Rr + 125r^2)s^4 - r^2(116R^2 + 84Rr + 16r^2)s^2 - r^3(4R + r)^3) + 8s^2(9s^2 + 6Rr + r^2)(s^2 + 10Rr + r^2)}{16s^2(9s^2 + 6Rr + r^2)^2} \overset{?}{<} \frac{1}{9}$$

$$\Leftrightarrow (216R + 1017r)s^4 + r(108R^2 + 516Rr + 130r^2)s^2 + 9r^2(4R + r)^3 \overset{?}{>} 0 \rightarrow \text{true}$$

$$\therefore \frac{p_a - m_a}{a} + \frac{p_b - m_b}{b} + \frac{p_c - m_c}{c} < \frac{1}{3} \ \forall \Delta ABC \ (QED)$$