ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$n_a + n_b + n_c \le p_a + p_b + p_c + \frac{4}{5}(max\{a,b,c\} - min\{a,b,c\})$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$n_a - p_a \le \frac{s|b-c|}{2s+a} \text{ and analogs}$$

(Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba – 52; published at www.ssmrmh.ro)

$$\therefore \sum_{cyc} n_a - \sum_{cyc} p_a \le \sum_{cyc} \frac{s|b-c|}{2s+a} \overset{CBS}{\le} s.\sqrt{\sum_{cyc}\frac{(b-c)^2}{2s+a}}.\sqrt{\sum_{cyc}\frac{1}{2s+a}}$$

$$= s.\sqrt{\frac{1}{2s(9s^2+6Rr+r^2)}\sum_{cyc}\left((8s^2-2sa+bc)\left(\sum_{cyc}a^2 - a^2 - 2bc\right)\right)}.\sqrt{\frac{21s^2+4Rr+r^2}{2s(9s^2+6Rr+r^2)}}$$

$$= s.\sqrt{\frac{\begin{matrix}2(s^2-4Rr-r^2)(24s^2-4s^2+s^2+4Rr+r^2) - \\ 16s^2(s^2-4Rr-r^2)+4s^2(s^2-6Rr-3r^2) - \\ 8Rrs^2 - 16s^2(s^2+4Rr+r^2)+48Rrs^2 - \\ 2((s^2+4Rr+r^2)^2-16Rrs^2)\end{matrix}}{2s(9s^2+6Rr+r^2)}}.\sqrt{\frac{21s^2+4Rr+r^2}{2s(9s^2+6Rr+r^2)}}$$

$$\therefore \sum_{cyc} n_a - \sum_{cyc} p_a \overset{①}{\le} s.\sqrt{\frac{(3s^4-(32Rr+14r^2)s^2-r^2(4R+r)^2)(21s^2+4Rr+r^2)}{4s^2(9s^2+6Rr+r^2)^2}}$$

$$\text{and again, } \frac{4}{5}(max\{a,b,c\} - min\{a,b,c\}) = \frac{2}{5}\sum_{cyc}|b-c|$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 45; published at www.ssmrmh.ro)

$$= \frac{2}{5}.\sqrt{\sum_{cyc}(b-c)^2 + 2\sum_{cyc}(|c-a||a-b|)} \overset{\text{Triangle Inequality}}{\ge}$$

$$\frac{2}{5}.\sqrt{2(s^2-12Rr-3r^2)+2\left|\sum_{cyc}(c-a)(a-b)\right|}$$

$$= \frac{2}{5}.\sqrt{2(s^2-12Rr-3r^2)+2\left|\sum_{cyc}a^2 - \sum_{cyc}ab\right|} = \frac{2}{5}.\sqrt{4(s^2-12Rr-3r^2)}$$

$$\overset{②}{=} \frac{4}{5}(max\{a,b,c\} - min\{a,b,c\}) \therefore ① \text{ and } ② \Rightarrow \text{it suffices to prove :}$$

$$\frac{(3s^4-(32Rr+14r^2)s^2-r^2(4R+r)^2)(21s^2+4Rr+r^2)}{4(9s^2+6Rr+r^2)^2}\overset{?}{\le}\frac{16(s^2-12Rr-3r^2)}{25}$$

$$\Leftrightarrow 3609s^6-(38796Rr+7125r^2)-r^2(69040R^2+27392Rr+2517r^2)s^2-$$

$$r^3(26048R^3+14928R^2r+2772Rr^2+167r^3)\overset{(\bullet)}{\ge}0\ and\ \because P=$$

$$3609(s^2-16Rr+5r^2)^3+4r(33609Rr-15315r)(s^2-16Rr+5r^2)^2+$$

$$8r^2\left(26R^2-Rr-6r^2+32(5707R^2-6249Rr+1326r^2)\right)\overset{\text{Gerretsen}}{\ge}0$$

$$\therefore \text{in order to prove }(\bullet),\text{it suffices to prove : LHS of }(\bullet)\overset{?}{\ge}P$$

$$\Leftrightarrow 26t^3+13t^2+4t-12+32(3632t^3-9359t^2+4487t-602)\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(26t^2+t+6+32(3632t^2-2093t+301)\right)\overset{?}{\ge}0\to \text{true}\because t\overset{\text{Euler}}{\ge}2$$

$$\Rightarrow (\bullet)\text{ is true}\therefore n_a+n_b+n_c\le p_a+p_b+p_c+\frac{4}{5}(max\{a,b,c\}-\min\{a,b,c\})$$

$$\forall\,\Delta\,ABC,''='' \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$