ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\ge\frac{1}{5}\cdot\sqrt{\frac{64R}{r}+97}$$

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$$\frac{p_a^2-h_a^2}{h_a^2}\overset{\substack{\text{Fustei}\\\text{and}\\\text{Ben Ajiba}}}{=}\frac{s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-\left(s(s-a)-\frac{s(s-a)}{a^2}(b-c)^2\right)}{h_a^2}$$

$$=\frac{\frac{4s^4}{a^2(2s+a)^2}}{\frac{4r^2s^2}{a^2}}.(b-c)^2\therefore\frac{p_a^2}{h_a^2}\overset{(1)}{=}1+\frac{s^2}{r^2}\cdot\frac{(b-c)^2}{(2s+a)^2}\text{ and analogs}\therefore\sum_{cyc}\frac{p_a^2}{h_a^2}\overset{(2)}{=}3+$$

$$\frac{s^2}{r^2}\cdot\left(\left(\sum_{cyc}\frac{b-c}{2s+a}\right)^2-\frac{2}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc}\big((b-c)(c-a)(2s+c)\big)\right)$$

$$\text{Now,}\left(\sum_{cyc}\frac{b-c}{2s+a}\right)^2=\frac{1}{4s^2(9s^2+6Rr+r^2)^2}\cdot\left(\sum_{cyc}\big((b-c)(8s^2-2sa+bc)\big)\right)^2$$

$$=\frac{1}{4s^2(9s^2+6Rr+r^2)^2}\cdot\left(\sum_{cyc}b^2c-\sum_{cyc}bc^2\right)^2$$

$$=\frac{(\sum_{cyc}a^2)(\sum_{cyc}a^2b^2)-15a^2b^2c^2-2abc\sum_{cyc}a^3-2\sum_{cyc}a^3b^3+2abc(\sum_{cyc}a)(\sum_{cyc}ab)}{4s^2(9s^2+6Rr+r^2)^2}$$

$$\overset{(*)}{=}\left(\sum_{cyc}\frac{b-c}{2s+a}\right)^2\text{ and also,}-\frac{2}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc}\big((b-c)(c-a)(2s+c)\big)\overset{(**)}{=}$$

$$-2.\frac{2s(\sum_{cyc}ab-\sum_{cyc}a^2)+2s(\sum_{cyc}ab)-\sum_{cyc}a^3-24Rrs}{2s(9s^2+6Rr+r^2)}\text{ and since : }\sum_{cyc}ab=$$

$$s^2+4Rr+r^2,\sum_{cyc}a^2=2(s^2-4Rr-r^2),\sum_{cyc}a^2b^2=(s^2+4Rr+r^2)^2-16Rrs^2,$$

$$\sum_{cyc}a^3=2(s^2-6Rr-3r^2),\sum_{cyc}a^3b^3=(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)$$

$$\therefore (2),(*),(**)\Rightarrow\sum_{cyc}\frac{p_a^2}{h_a^2}\boxed{\overset{(m)}{=}}3+\frac{18s^6-(168Rr+125r^2)s^4-r^2(116R^2+84Rr+16r^2)s^2-r^3(4R+r)^3}{r^2(9s^2+6Rr+r^2)^2}$$

$$\text{and again}, ① \Rightarrow \sum_{cyc} \frac{p_b p_c}{h_b h_c} = \sum_{cyc} \sqrt{\left(1 + \frac{s^2}{r^2} \cdot \frac{(c-a)^2}{(2s+b)^2}\right) \cdot \left(1 + \frac{s^2}{r^2} \cdot \frac{(a-b)^2}{(2s+c)^2}\right)}$$

$$\overset{\text{Reverse CBS}}{\geq} 3 + \frac{s^2}{r^2} \cdot \frac{1}{2s(9s^2+6Rr+r^2)} \cdot \sum_{cyc} \left(|c-a||a-b|(2s+a)\right) \overset{\text{Triangle Inequality}}{\geq}$$

$$3 + \frac{s^2}{r^2} \cdot \frac{1}{2s(9s^2+6Rr+r^2)} \cdot \left(2s\left|\sum_{cyc}(c-a)(a-b)\right| + \left|\sum_{cyc} a(c-a)(a-b)\right|\right)$$

$$= 3 + \frac{s^2\left((s^2-12Rr-3r^2)+(2Rr-4r^2)\right)}{r^2(9s^2+6Rr+r^2)} \therefore 2\sum_{cyc} \frac{p_b p_c}{h_b h_c} + \sum_{cyc} \frac{p_a^2}{h_a^2} \geq 9 +$$

$$\frac{s^2(9s^2+6Rr+r^2)(2s^2-20Rr-14r^2) + 18s^6 - (168Rr+125r^2)s^4 - r^2(116R^2+84Rr+16r^2)s^2 - r^3(4R+r)^3}{r^2(9s^2+6Rr+r^2)^2} \overset{?}{\geq} \frac{64R+97r}{25r}$$

$$\Leftrightarrow 900s^6 - (13584Rr - 4143r^2)s^4 - r^2(12812R^2 - 7972Rr - 1554r^2)s^2 -$$

$$r^3(3904R^3 - 2640R^2r - 1172Rr^2 - 103r^3) \overset{(\bullet)}{\geq} 0; \because P = 900(s^2 - 16Rr + 5r^2)^3$$
$$+(29616Rr - 9357r^2)(s^2 - 16Rr + 5r^2)^2 +$$

$$4r^2(60925R^2 - 38903Rr + 6906r^2)\left(s^2 - 16Rr + 5r^2 - \frac{r^2(R-2r)}{R-r}\right) \overset{\substack{\text{Gerretsen} \\ \text{and Rouche} \\ +\text{Euler}}}{\geq} 0$$

$$\left(\begin{array}{c} \because (R-r)(s^2-16Rr+5r^2) \overset{\text{Rouche}}{\geq} \\ (R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right) \\ = (R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2 + r^2\right) \geq r^2(R-2r) \left(\because R-2r \overset{\text{Euler}}{\geq} 0\right) \\ \Rightarrow s^2 \geq 16Rr - 5r^2 + \dfrac{r^2(R-2r)}{R-r} \end{array}\right)$$

$$\therefore \text{in order to prove } (\bullet), \text{it suffices to prove : LHS of } (\bullet) \overset{?}{\geq} P$$

$$\Leftrightarrow (t-2)(3375t^2 - 1869t + 302) \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \left(\sum_{cyc} \frac{p_a}{h_a}\right)^2 \geq \frac{64R+97r}{25r} \Rightarrow \frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \geq \frac{1}{5} \cdot \sqrt{\frac{64R}{r} + 97} \ \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$