ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\ge\frac{3}{2}\cdot\sqrt{\frac{R}{r}+2}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2-h_a^2}{h_a^2}=\frac{s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)}{a^2}(b-c)^2\right)}{h_a^2}$$

$$=\frac{\frac{4s^2-4sa+a^2}{4a^2}}{\frac{4r^2s^2}{a^2}}.(b-c)^2=\frac{(2s-a)^2(b-c)^2}{16r^2s^2}=\frac{(b+c)^2(b-c)^2}{16r^2s^2}=\frac{(b^2-c^2)^2}{16r^2s^2}$$

$$\therefore\frac{m_a}{h_a}=\sqrt{1+\frac{(b^2-c^2)^2}{16r^2s^2}}\text{ and analogs }\therefore 2\sum_{cyc}\left(\frac{m_b}{h_b}.\frac{m_c}{h_c}\right)=$$

$$2\sum_{cyc}\sqrt{\left(1+\frac{(c^2-a^2)^2}{16r^2s^2}\right).\left(1+\frac{(a^2-b^2)^2}{16r^2s^2}\right)}\overset{\text{Reverse CBS}}{\ge}$$

$$2\sum_{cyc}\left(1+\frac{|c^2-a^2||a^2-b^2|}{16r^2s^2}\right)=6+\sum_{cyc}\frac{|a^2-b^2|.\left(|b^2-c^2|+|c^2-a^2|\right)}{16r^2s^2}$$

$$\overset{\text{Triangle Inequality}}{\ge}6+\sum_{cyc}\frac{|a^2-b^2|.\left(|b^2-c^2+c^2-a^2|\right)}{16r^2s^2}=6+\sum_{cyc}\frac{(a^2-b^2)^2}{16r^2s^2}$$

$$=6+\frac{2\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)-2\sum_{cyc}a^2b^2}{16r^2s^2}=6+\frac{2\sum_{cyc}a^2b^2-32r^2s^2}{16r^2s^2}\text{ and so,}$$

$$\sum_{cyc}\frac{m_a^2}{h_a^2}+2\sum_{cyc}\left(\frac{m_b}{h_b}.\frac{m_c}{h_c}\right)\ge\sum_{cyc}\frac{a^2\left(2\sum_{cyc}a^2-3a^2\right)}{16r^2s^2}+\frac{2\sum_{cyc}a^2b^2+64r^2s^2}{16r^2s^2}$$

$$=\frac{2\left(\sum_{cyc}a^2\right)^2-3\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)+2\sum_{cyc}a^2b^2+64r^2s^2}{16r^2s^2}$$

$$=\frac{2(s^2-4Rr-r^2)^2-\left((s^2+4Rr+r^2)^2-16Rrs^2\right)+28r^2s^2}{4r^2s^2}\overset{?}{\ge}\frac{9}{4}.\frac{R+2r}{r}$$

$$\Leftrightarrow s^4-(17Rr-4r^2)s^2+r^2(4R+r)^2\overset{?}{\underset{①}{\ge}}0\text{ and }\because(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0$$

$$\therefore\text{ in order to prove ①, it suffices to prove : LHS of ①}\overset{?}{\ge}(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(5R-2r)s^2\overset{?}{\underset{②}{\ge}}r(80R^2-56Rr+8r^2)\ \&\text{ now, }(R-r)(s^2-16Rr+5r^2)\overset{\text{Rouche}}{\ge}$$

$$(R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right)$$

$$= (R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2 + r^2\right) \ge r^2(R-2r)\ \left(\because R-2r \overset{\text{Euler}}{\ge} 0\right)$$

$$\Rightarrow s^2 \ge 16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} \therefore (5R-2r)s^2 \ge$$

$$(5R-2r)\left(16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r}\right) \overset{?}{\ge} r(80R^2 - 56Rr + 8r^2)$$

$$\Leftrightarrow r^2(R-2r)(4R-r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\therefore \frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \ge \frac{3}{2}.\sqrt{\frac{R}{r}+2}\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$