ROMANIAN MATHEMATICAL MAGAZINE

In Δ ABC with $n_a, n_b, n_c \to$ Nagel cevians holds :

$$\frac{n_a + n_b + n_c}{h_a + h_b + h_c} \ge \frac{3R}{R + 4r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{n_a + n_b + n_c}{h_a + h_b + h_c} \ge \frac{\sqrt{9s^2 - 80Rr - 2r^2}}{s^2 + 4Rr + r^2}.2R$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 74;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right) \overset{?}{\ge} \frac{3R}{R + 4r}$$

$$\overset{\text{squaring}}{\Leftrightarrow} -9s^4 + (36R^2 + 216Rr + 558r^2)s^2 -$$

$$r(320R^3 + 2712R^2r + 5256Rr^2 + 137r^3) \underset{(*)}{\overset{?}{\ge}} 0$$

$$\text{Now, since } P = -9(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3) \overset{\text{Double-Rouche}}{\ge} 0$$

$$\therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\ge} P$$

$$\Leftrightarrow (9R + 144r)s^2 + 64R^3 - 570R^2r - 1287Rr^2 - 32r^3 \underset{(**)}{\overset{?}{\ge}} 0$$

$$\text{Again, LHS of } (**) \overset{\text{Gerretsen}}{\ge} (9R + 144r)(16Rr - 5r^2) +$$

$$64R^3 - 570R^2r - 1287Rr^2 - 32r^3 \overset{?}{\ge} 0 \Leftrightarrow 32R^3 - 213R^2r + 486Rr^2 - 376r^3 \overset{?}{\ge} 0$$

$$\Leftrightarrow (R - 2r)(32R^2 - 149Rr + 188r^2) \overset{?}{\ge} 0 \to \text{true} \because R - 2r \overset{\text{Euler}}{\ge} 0 \text{ and}$$

$$\text{discriminant of } 32R^2 - 149Rr + 188r^2 = 149^2.r^2 - 128.188.r^2 = -1863r^2 < 0$$

$$\Rightarrow 32R^2 - 149Rr + 188r^2 > 0 \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{n_a + n_b + n_c}{h_a + h_b + h_c} \ge \frac{3R}{R + 4r}$$

$$\forall \Delta \text{ ABC}, ''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$