ROMANIAN MATHEMATICAL MAGAZINE

In Δ ABC with $n_a, n_b, n_c \to$ Nagel cevians then :

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c} \geq \sqrt{\frac{s^2-12Rr}{r^2}+6} \geq \sqrt{\frac{4R}{r}+1}$$

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$$\sum_{cyc}\frac{n_a^2}{h_a^2} \overset{\text{Bogdan Fustei}}{=} \sum_{cyc}\frac{a^2s^2-\frac{r}{R}.s^2\sec^2\frac{A}{2}.16R^2\cos^2\frac{A}{2}\sin^2\frac{A}{2}}{4r^2s^2}$$

$$= \frac{s^2-4Rr-r^2-8Rr.\frac{2R-r}{2R}}{2r^2} \therefore \sum_{cyc}\frac{n_a^2}{h_a^2} \overset{①}{=} \frac{s^2-12Rr+3r^2}{2r^2}$$

$$\text{Now, } \frac{n_a^2-h_a^2}{h_a^2} = \frac{s(s-a)+\frac{s}{a}(b-c)^2-\left(s(s-a)-\frac{s(s-a)}{a^2}(b-c)^2\right)}{h_a^2}$$

$$= \frac{\frac{s^2}{a^2}}{\frac{4r^2s^2}{a^2}}.(b-c)^2 = \frac{(b-c)^2}{4r^2} \therefore \frac{n_a}{h_a} = \sqrt{1+\frac{(b-c)^2}{4r^2}} \text{ and analogs } \therefore 2\sum_{cyc}\left(\frac{n_b}{h_b}.\frac{n_c}{h_c}\right) =$$

$$2\sum_{cyc}\sqrt{\left(1+\frac{(c-a)^2}{4r^2}\right).\left(1+\frac{(a-b)^2}{4r^2}\right)} \overset{\text{Reverse CBS}}{\geq} 2\sum_{cyc}\left(1+\frac{|c-a||a-b|}{4r^2}\right)$$

$$= 6+\sum_{cyc}\frac{|a-b|.(|b-c|+|c-a|)}{4r^2} \overset{\text{Triangle Inequality}}{\geq}$$

$$6+\sum_{cyc}\frac{|a-b|.(|b-c+c-a|)}{4r^2} = 6+\sum_{cyc}\frac{(a-b)^2}{4r^2} = 6+\frac{1}{2r^2}.\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$\therefore 2\sum_{cyc}\left(\frac{n_b}{h_b}.\frac{n_c}{h_c}\right) \overset{②}{\geq} 6+\frac{s^2-12Rr-3r^2}{2r^2} \therefore \sum_{cyc}\frac{n_a^2}{h_a^2}+2\sum_{cyc}\left(\frac{n_b}{h_b}.\frac{n_c}{h_c}\right) \overset{\text{via } ①+②}{\geq}$$

$$\frac{s^2-12Rr+3r^2}{2r^2}+6+\frac{s^2-12Rr-3r^2}{2r^2} = \frac{s^2-12Rr}{r^2}+6$$

$$\therefore \left(\sum_{cyc}\frac{n_a}{h_a}\right)^2 \geq \frac{s^2-12Rr}{r^2}+6 \Rightarrow \sum_{cyc}\frac{n_a}{h_a} \geq \sqrt{\frac{s^2-12Rr}{r^2}+6} \overset{\text{Gerretsen}}{\geq}$$

$$\sqrt{\frac{16Rr-5r^2-12Rr}{r^2}+6} = \sqrt{\frac{4R}{r}+1} \text{ and so,}$$

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c} \geq \sqrt{\frac{s^2-12Rr}{r^2}+6} \geq \sqrt{\frac{4R}{r}+1}\ \forall\,\Delta\,ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)