ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c

$\rightarrow$ Nagel cevians, the following relationship holds :

$$n_a + n_b + n_c + \sqrt{3}(AI + BI + CI) \geq \left(2 + \sqrt{3}\right)s$$

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Solution by Soumava Chakraborty-Kolkata-India

$$AI = \frac{r}{\sin\frac{A}{2}} = \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2} = \frac{s-a}{\cos\frac{A}{2}} \Rightarrow \cos\frac{A}{2} = \frac{s-a}{AI}$$

$$\therefore \tan\frac{A}{4} = \frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}} = \frac{1-\frac{s-a}{AI}}{\frac{r}{AI}} = \frac{AI-(s-a)}{r}$$

$$\Rightarrow AI = s - a + r\tan\frac{A}{4} \text{ and analogs} \overset{\text{via summation}}{\Rightarrow} \sum AI = s + r\sum\tan\frac{A}{4} \rightarrow (1)$$

$$\text{Let } f(x) = \tan\frac{x}{4}\ \forall x \in (0,\pi) \text{ and then : } f''(x) = \frac{\tan\frac{x}{4}\sec^2\frac{x}{4}}{8} > 0 \Rightarrow f(x) \text{ is convex}$$

$$\text{Let } \tan\frac{\pi}{12} = m \therefore \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{2m}{1-m^2} \Rightarrow m^2 + 2\sqrt{3}m - 1 = 0$$

$$\Rightarrow m = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = 2 - \sqrt{3} \Rightarrow \tan\frac{\pi}{12} = 2 - \sqrt{3}$$

$$\therefore \sum AI \overset{\text{via (1)}}{=} s + r\sum\tan\frac{A}{4} \overset{\text{Jensen}}{\geq} s + 3r\tan\frac{\pi}{12} = s + 3r\left(2-\sqrt{3}\right)$$

$$\Rightarrow n_a + n_b + n_c + \sqrt{3}(AI + BI + CI)$$

$$\overset{\text{Mohamed Amine Ben Ajiba}}{\geq} 2s - 3\left(2\sqrt{3}-3\right)r + \sqrt{3}\left(s + 3r\left(2-\sqrt{3}\right)\right)$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 77,\\ \text{published at www.ssmrmh.ro}\end{array}\right)$$

$$= 9r + 2\left(s - 3\sqrt{3}r\right) + \sqrt{3}\left(6r + s - 3\sqrt{3}r\right) = 9r + 2s - 6\sqrt{3}r + 6\sqrt{3}r + \sqrt{3}s - 9r$$

$$= \left(2+\sqrt{3}\right)s \therefore n_a + n_b + n_c + \sqrt{3}(AI + BI + CI) \geq \left(2+\sqrt{3}\right)s\ \forall\ \Delta\ ABC,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$