ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{x}{y+z}w_a^2+\frac{y}{z+x}w_c^2+\frac{z}{x+y}w_c^2\ge 27r^2-\frac{s^2}{2}$$

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Solution by Tapas Das-India

$$\sum w_aw_b\overset{w_a\ge h_a}{\ge}\sum h_ah_b=4F^2\sum\frac{1}{ab}=4F^2\frac{a+b+c}{abc}=4F^2\frac{2s}{4RF}=\frac{2Fs}{R}=$$

$$=\frac{2rs^2}{R}\overset{s^2\ge\frac{27Rr}{2}}{\ge}\frac{2r}{R}\cdot\frac{27Rr}{2}=27r^2\ (1)$$

$$\frac{x}{y+z}w_a^2+\frac{y}{z+x}w_c^2+\frac{z}{x+y}w_c^2=$$

$$=\sum\frac{x}{y+z}w_a^2=\sum\left(\frac{x}{y+z}+1-1\right)w_a^2=(x+y+z)\sum\frac{w_a^2}{y+z}-\sum w_a^2\overset{CBS}{\ge}$$

$$\ge(x+y+z)\frac{(\sum w_a)^2}{2(x+y+z)}-\sum w_a^2=\frac{(\sum w_a)^2}{2}-\sum w_a^2=$$

$$=\frac{\sum w_a^2+2\sum w_aw_b}{2}-\sum w_a^2=$$

$$=\sum w_aw_b-\frac{1}{2}\sum w_a^2\overset{w_a\le\sqrt{s(s-a)}\ \&\ (1)}{\ge}27r^2-\frac{1}{2}\sum s(s-a)=27r^2-\frac{s^2}{2}$$

Equality holds for an equilateral triangle and x=y=z.