ROMANIAN MATHEMATICAL MAGAZINE

Let ABC be a triangle. $[AD]$ its interior bisector. r_1 and r_2 are inradius of the triangle ABD and ADC respectively .

Prove that:

$$r_1 < r_2 \Rightarrow \ b > c$$

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AD is internal bisector :

$$BD: DC = c: b, BD = \frac{ac}{b+c}, DC = \frac{ab}{b+c}, AD = \frac{2bc}{b+c}\cos\frac{A}{2}, \frac{[ABD]}{[ADC]} = \frac{BD}{DC} = \frac{c}{b}$$

$$r_1 = \frac{[ABD]}{\frac{1}{2}(AB + BD + AD)} = \frac{2[ABD]}{c + \frac{ac}{b+c} + \frac{2bc}{b+c}\cos\frac{A}{2}},$$

$$r_2 = \frac{[ADC]}{AD + DC + AC} = \frac{2[ADC]}{\frac{2bc}{b+c}\cos\frac{A}{2} + \frac{ab}{b+c} + b}$$

$$\frac{r_1}{r_2} = \frac{[ABD]}{[ADC]}\frac{\frac{2bc}{b+c}\cos\frac{A}{2} + \frac{ab}{b+c} + b}{c + \frac{ac}{b+c} + \frac{2bc}{b+c}\cos\frac{A}{2}} = \frac{c}{b}\frac{\frac{2bc}{b+c}\cos\frac{A}{2} + \frac{ab}{b+c} + b}{c + \frac{ac}{b+c} + \frac{2bc}{b+c}\cos\frac{A}{2}} = \frac{a + b + c + 2c\cos\frac{A}{2}}{a + b + c + 2b\cos\frac{A}{2}}$$

$$r_1 < r_2 \Rightarrow \frac{r_1}{r_2} < 1$$

$$\Rightarrow \frac{a + b + c + 2c\cos\frac{A}{2}}{a + b + c + 2b\cos\frac{A}{2}} < 1 \Rightarrow$$

$$a + b + c + 2c\cos\frac{A}{2} < a + b + c + 2b\cos\frac{A}{2}$$

$$\Rightarrow 2c\cos\frac{A}{2} < 2b\cos\frac{A}{2} \ \Rightarrow \ c < b$$