ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{3}{\sqrt{R}} \le \sum \frac{1}{\sqrt{w_a - r}} \le \frac{3}{\sqrt{2r}}$$

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Solution by Tapas Das-India

$$\sum w_a = \sum \frac{2bc}{b+c}\cos\frac{A}{2} \overset{AM-GM}{\le} \sum \frac{2bc}{2\sqrt{bc}}\cos\frac{A}{2} =$$

$$= \sum \sqrt{bc}\cos\frac{A}{2} \overset{CBS}{\le} \sqrt{(bc+ca+ab)\left(\sum \cos^2\frac{A}{2}\right)} =$$

$$= \sqrt{(s^2+r^2+4Rr)\left(\frac{4R+r}{2R}\right)} \overset{Gerretsen}{\le}$$

$$\le \sqrt{(4R^2+4Rr+3r^2+r^2+4Rr)\left(\frac{4R+r}{2R}\right)} = \sqrt{\frac{4(R+r)^2(4R+r)}{2R}} \quad (1)$$

$$\sum \frac{1}{\sqrt{w_a - r}} = \sum \frac{1^{\frac{3}{2}}}{\sqrt{w_a - r}} \overset{Radon}{\ge} \frac{(3)^{\frac{3}{2}}}{\sqrt{\sum w_a - 3r}}$$

$$We\ need\ to\ show:$$

$$\frac{(3)^{\frac{3}{2}}}{\sqrt{\sum w_a - 3r}} \ge \frac{3}{\sqrt{R}}\ or\ 3R \overset{squaring}{\ge} \sum w_a - 3r\ or\ 3(R+r) \overset{(1)}{\ge} \sqrt{\frac{4(R+r)^2(4R+r)}{2R}}$$

$$9(R+r)^2 \overset{squaring}{\ge} \frac{4(R+r)^2(4R+r)}{2R}\ or\ 9R \ge 2(4R+r)\ or\ R \ge 2r\ (Euler)$$

$$Lemma: w_a \ge r + \frac{r}{\sin\frac{A}{2}} (Mohamed\ Amine\ Ben\ Ajiba)$$

$$Proof: w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{bc\sin A}{(b+c)\sin\frac{A}{2}} = \frac{2F}{(b+c)\sin\frac{A}{2}} = \frac{(a+b+c)r}{(b+c)\sin\frac{A}{2}} =$$

$$= \frac{ar}{(b+c)\sin\frac{A}{2}} + \frac{r}{\sin\frac{A}{2}} \overset{Mollweide}{=} \frac{r}{\cos\frac{B-C}{2}} + \frac{r}{\sin\frac{A}{2}} \overset{\cos\frac{B-C}{2}\le 1}{\ge} r + \frac{r}{\sin\frac{A}{2}}$$

$$\sum \frac{1}{\sqrt{w_a - r}} \overset{w_a \ge r + \frac{r}{\sin\frac{A}{2}}}{\le} \sum \sqrt{\frac{\sin\frac{A}{2}}{r}} \overset{CBS}{\le} \frac{1}{\sqrt{r}}\sqrt{3\sum \sin\frac{A}{2}} \overset{Jensen}{\le} \frac{1}{\sqrt{r}}\sqrt{3.3\sin\frac{\pi}{6}} = \frac{3}{\sqrt{2r}}$$

Equality holds for an equilateral triangle.