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In ΔABC he following relationship holds:

$$\left(3+\sum\frac{a^2+b^2}{c^2}\right)\left(\sum\sec^2\frac{A}{2}\right)\le\frac{9}{4}\left(\frac{R}{r}\right)^4$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$\left(3+\sum\frac{a^2+b^2}{c^2}\right)=\sum\left(\frac{a^2+b^2}{c^2}+1\right)=\sum\frac{a^2+b^2+c^2}{c^2}$$
$$=\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)\overset{Leibniz\,\&\,Steining}{\le}\frac{9R^2}{4r^2}\ (1)$$

$$\sum\sec^2\frac{A}{2}=3+\sum\tan^2\frac{A}{2}=3+\left(\frac{4R+r}{s}\right)^2-2=1+\frac{(4R+r)^2}{s^2}\le$$
$$\overset{s^2\ge3r(4R+r)}{\le}1+\frac{(4R+r)^2}{3r(4R+r)}=1+\frac{4R+r}{3r}=\frac{4}{3}+\frac{4R}{3r}=$$
$$=\frac{4}{3}\left(1+\frac{R}{r}\right)\overset{Euler}{\le}\frac{4}{3}\left(\frac{R}{2r}+\frac{R}{r}\right)=\frac{2R}{r}\ (2)$$

$$Using\ (1)\,\&\,(2)we\ get:$$
$$\left(3+\sum\frac{a^2+b^2}{c^2}\right)\left(\sum\sec^2\frac{A}{2}\right)\le\frac{9R^2}{4r^2}.\frac{2R}{r}=\frac{9}{2}\left(\frac{R}{r}\right)^3\overset{Euler}{\le}\frac{9}{2}\left(\frac{R}{r}\right)^3.\frac{R}{2r}=\frac{9}{4}\left(\frac{R}{r}\right)^4$$

Equality holds for an equilateral triangle.